Solutions to homework #1

- **6.1.0 a)** False. You can use a divergent $p$ series as a counterexample, or $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + \sqrt{k+1}}$, which I showed in class was a diverging telescoping series.

- **6.1.0 b)** False. As a counterexample, take any convergent series whose terms are nonzero, say $\sum_{k=1}^{\infty} \frac{1}{k^p}$, which I showed in class was a diverging telescoping series.

- **6.1.0 c)** True. If $\sum_{k} a_k$ converges then $\sum_{k} b_k = \sum_{k} (a_k + b_k)$ is the sum of two convergent series, hence convergent. Similarly, if $\sum_{k} b_k$ converges so does $\sum_{k} a_k$. Thus the series both converge or both diverge.

- **6.1.0 d)** True: Look at the partial sums:

$$s_n = \sum_{k=1}^{n} (a_k - a_{k+2})$$

$$= (a_1 - a_3) + (a_2 - a_4) + (a_3 - a_5) + \cdots + (a_{n-1} - a_{n+1}) + (a_n - a_{n+2})$$

$$= a_1 + a_2 - a_{n+1} - a_{n+2}.$$ 

so the limit of the partial sums exists and equals $a_1 + a_2 - 2a$, which is therefore the sum of the series.

- **6.1.1 d)** This is the sum of two convergent geometric series, hence convergent:

$$\sum_{k=0}^{\infty} \frac{5^{k+1} + (-3)^k}{7^{k+2}} = \sum_{k=0}^{\infty} \left( \frac{5}{49} \right)^k \left( \frac{5}{7} \right)^k + \sum_{k=0}^{\infty} \left( \frac{1}{49} \right)^k \left( \frac{3}{7} \right)^k$$

$$= \frac{5/49}{1 - (5/7)} + \frac{1/49}{1 + (3/7)}$$

$$= \frac{5}{50 - 35} + \frac{1}{49 + 21}$$

$$= \frac{5}{14} + \frac{1}{70} = \frac{26}{70}$$

- **6.1.2 c)** This telescopes.

$$\sum_{k=2}^{N} \log \left( \frac{k(k+2)}{(k+1)^2} \right) = \sum_{k=2}^{N} (\log (k) + \log (k+2) - 2 \log (k+1))$$

$$= (\log 2 + \log 4 - 2 \log 3) + (\log 3 + \log 5 - 2 \log 4)$$

$$+ (\log 4 + \log 6 - 2 \log 5) + \cdots$$

$$+ (\log (N-1) + \log (N+1) - 2 \log (N))$$

$$+ (\log (N) + \log (N+2) - 2 \log (N+1))$$

$$= \log 2 - \log 3 + \log (N+2) - \log (N+1)$$

$$= \log \left( \frac{2(N+2)}{3(N+1)} \right) \to \log \left( \frac{2}{3} \right).$$
If you’re careful, you will prove the formula for the telescope by induction, since it’s pretty complicated, but I didn’t require it.

- 6.2.1 c) This is similar to an example in lecture. Since $p > 1$, we can write $p$ as $1 + 2a$ for some $a > 0$. Then

$$\frac{\log k}{k^p} = \frac{1}{k^{1+a}} \frac{\log k}{k^a}.$$ 

Since

$$\lim_{k \to \infty} \frac{\log k}{k^a} = \lim_{k \to \infty} \frac{1/k}{ak^{a-1}} = \lim_{k \to \infty} \frac{1}{ak^a} = 0,$$

(since $a > 0$), for large enough $k$, $\frac{\log k}{k^a} < 1$. We can then say that $\sum_{k=1}^{\infty} \frac{\log k}{k^a}$ converges by the basic comparison test with the convergent $p$ series $\sum \frac{1}{k^a}$. You can also use the limit comparison test again comparing with $\sum \frac{1}{k^{a'}}$. Here they’ll be either in case 2 or 3 of the LCT (depending on which term you put on top), so you have to be careful.

- 6.2.7 Since $\sum a_k$ converges, $\lim_{k \to \infty} a_k = 0$. Thus for sufficiently large $k$, $0 \leq a_k < 1$, hence for sufficiently large $k$, $0 \leq a_kb_k \leq b_k$ (using the facts that $a_k$ and $b_k$ are non-negative). Hence $\sum a_kb_k$ converges by BCT with $\sum b_k$. (You can also use the limit comparison test comparing $\sum a_kb_k$ and $\sum b_k$ (or $\sum a_k$ for that matter), but again you’ll be in case 2 or 3 and you have to be cautious.)

- 6.3.0 d) This is true. You can either use 6.2.7 (taking both sequences to be $\sum |a_k|$), or say that, since $\sum |a_k|$ converges, the individual terms must eventually be less than 1, and once that happens, $0 \leq a_k^2 \leq |a_k|$ and the result follows from the BCT.

- 6.3.2 g) Use the root test. The sequence of roots is

$$\left\{ \frac{3 - (-1)^k}{\pi} \right\} = \left\{ \frac{4}{\pi}, \frac{2}{\pi}, \frac{4}{\pi}, \cdots \right\}.$$ 

The limsup of the sequence of $k^{th}$ roots is $\frac{4}{\pi} > 1$, hence the series diverges. (You can also notice that every other term is larger than 1, hence the terms can’t be going to zero).

- 6.3.3 a) Since we can integrate $\int \frac{1}{x \log(x)^p} \, dx$ by a substitution, we’re tempted to use the integral test. First, though, we must check that the function $\frac{1}{x \log(x)^p}$ is decreasing. We have

$$\frac{d}{dx} (x \log^p(x))^{-1} = -(x \log^p x)^{-2} (\log^{p-1} x (\log x + p)),$$
which is negative for \( x > e^{-p} \). Thus the function is at least eventually
decreasing (which is all we really need). Therefore, it makes sense to look
at \( \int_{2}^{\infty} \frac{1}{x \log^p x} \, dx \). Letting \( u = \log x \), we have

\[
\int_{2}^{\infty} \frac{1}{x \log^p x} \, dx = \int_{\log 2}^{\infty} \frac{1}{u} \, du
\]

This converges if and only if \( p > 1 \), thus by the integral test the original
series converges if and only if \( p > 1 \) as well. Since the terms are positive,
the series converges absolutely for those \( p \) values.

- 6.3.3 d) First, consider the case that \( p > 0 \). The terms of this series are
positive, so convergence and absolute convergence are the same. Use the
limit comparison test with \( \sum \frac{1}{k^{p+1/2}} \):

\[
\lim_{k \to \infty} \frac{1/\left(\sqrt{k} \,(k^p - 1)\right)}{1/k^{p+1/2}} = \lim_{k \to \infty} \frac{k^{p+1/2}}{k^{p+1/2} - \sqrt{k}} = \lim_{k \to \infty} \frac{1}{1 - k^{-p}} = 1,
\]

so the given series and \( \sum \frac{1}{k^{p+1/2}} \) behave the same way. If \( p + \frac{1}{2} > 1 \), the
series converges, and if \( p + \frac{1}{2} \leq 1 \), the series diverges. At this point we
know that the series converges if \( p > \frac{1}{2} \) and diverges if \( 0 < p \leq \frac{1}{2} \). Next,
the case \( p = 0 \) doesn’t make sense, since no term in the series is defined
(we’re always dividing by zero). Finally, to consider the case that \( p < 0 \).
Then all the terms in the series are negative, and looking at the series of
absolute values, we consider

\[
\sum \frac{1}{\sqrt{k} \,(1 - k^p)}
\]

Use the limit comparison test with \( \sum \frac{1}{\sqrt{k}} \):

\[
\lim_{k \to \infty} \frac{1/\sqrt{k}}{1/\left(\sqrt{k} \,(1 - k^p)\right)} = \lim_{k \to \infty} (1 - k^p) = 1.
\]

Since \( \sum \frac{1}{\sqrt{k}} \) diverges, the series we’re looking at diverges as well. Putting
all the cases together, we get convergence for \( p > \frac{1}{2} \), and divergence for
\( p \leq \frac{1}{2} \) (although for \( p = 0 \) we should note that the series is not defined.)