

Answers to exam 1, version A

1. (a) See definition (2), p. 102 of our text.

(b) Let $\varepsilon > 0$ be given. Set δ to be $\sqrt{\frac{\varepsilon}{2}}$. Then $0 < |x - 2| < \sqrt{\frac{\varepsilon}{2}}$ implies $|x - 2| < \sqrt{\frac{\varepsilon}{2}}$, thus $|(x - 2)^2| < \frac{\varepsilon}{2}$, implying $|2x^2 - 8x + 8| < \varepsilon$. This is $|2x^2 - 8x - (-8)| < \varepsilon$, therefore $\lim_{x \rightarrow 2} 2x^2 - 8x = -8$.

2.

(a)

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x + 1}{\sqrt{3x^2 + 2x}} &= \lim_{x \rightarrow \infty} \frac{(x + 1) \frac{1}{x}}{\sqrt{3x^2 + 2x} \left(\frac{1}{x}\right)} \\ &= \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{\sqrt{(3x^2 + 2x) \left(\frac{1}{x^2}\right)}} \\ &= \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{\sqrt{3 + \frac{2}{x}}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}. \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt{x + 7} - 3}{x - 2} &= \lim_{x \rightarrow 2} \frac{\sqrt{x + 7} - 3}{x - 2} \left(\frac{\sqrt{x + 7} + 3}{\sqrt{x + 7} + 3} \right) \\ &= \lim_{x \rightarrow 2} \frac{x + 7 - 9}{(x - 2)(\sqrt{x + 7} + 3)} \\ &= \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(\sqrt{x + 7} + 3)} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x + 7} + 3} = \frac{1}{6}. \end{aligned}$$

(c) For $x < \frac{1}{2}$, $2x - 1 < 0$, so $|2x - 1| = -(2x - 1)$ for those x 's. Then

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{2}^-} \frac{|2x - 1|}{2x^2 + x - 1} &= \lim_{x \rightarrow \frac{1}{2}^-} \frac{-(2x - 1)}{2x^2 + x - 1} \\ &= \lim_{x \rightarrow \frac{1}{2}^-} \frac{-(2x - 1)}{(2x - 1)(x + 1)} \\ &= \lim_{x \rightarrow \frac{1}{2}^-} \frac{-1}{x + 1} = -\frac{2}{3}. \end{aligned}$$

3. From the formula for component, we need

$$\frac{\langle 3, 4 \rangle \cdot \langle -1, x \rangle}{|\langle 3, 4 \rangle|} = 2,$$

so

$$\frac{-3 + 4x}{5} = 2,$$

which gives $x = \frac{13}{4}$.

4. For x to be in the domain of g , we need $9 - 4x^2 \geq 0$, which is $x^2 \leq \frac{9}{4}$, whose solution is $-\frac{3}{2} \leq x \leq \frac{3}{2}$, so the domain is $[-\frac{3}{2}, \frac{3}{2}]$. For the range, notice that $9 - 4x^2$ must be a positive number less than or equal to 9, therefore we get numbers out of g from 0 to $3 = \sqrt{9}$. Therefore the range is $[0, 3]$.

5.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{1}{h} (\vec{r}(1+h) - \vec{r}(1)) &= \lim_{h \rightarrow 0} \frac{1}{h} (\langle (1+h)^2 + 3(1+h), 2(1+h) \rangle - \langle 4, 2 \rangle) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (\langle h^2 + 5h, 2h \rangle) \\ &= \lim_{h \rightarrow 0} \langle h + 5, 2 \rangle = \langle 5, 2 \rangle.\end{aligned}$$

6. It's wrong to multiply the inequality by $x+1$ without considering the sign of $x+1$: if $x+1 < 0$, the direction of the inequality flips. Here's a correct way to do it: $\frac{x}{x+1} > 2$ is true if $\frac{x}{x+1} - 2 > 0$, which is the same as $\frac{x-2(x+1)}{x+1} > 0$, i.e. $\frac{-x-2}{x+1} > 0$, or $\frac{x+2}{x+1} < 0$ (notice that the inequality flipped when I multiplied by -1). This is true only when $x+1$ and $x+2$ have different signs, which occurs only for $-2 < x < -1$.
7. (a) No. The particle's x coordinate equals 5 when $t = 2$, but at that time, the y coordinate is $6 \cdot 2 - 4 = 8$, not 2.
(b) The object's path is $x = t + 3$, $y = 6t - t^2$. Solve the first equation for t to get $t = x - 3$. Plug that into the second equation to find that $y = 6(x - 3) - (x - 3)^2$, which can be simplified to $y = -x^2 + 12x - 27$.
8. Since $\sin^2 \theta + \cos^2 \theta = 1$, we must have $\frac{4}{9} + \cos^2 \theta = 1$, so $\cos^2 \theta = \frac{5}{9}$. Since $\theta \in (\frac{\pi}{2}, \pi)$, $\cos \theta$ must be negative, therefore $\cos \theta = -\frac{\sqrt{5}}{3}$. Finally, $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{2/3}{-\sqrt{5}/3} = -\frac{2}{\sqrt{5}} = -\frac{2}{5}\sqrt{5}$.
9. We have $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 2x - 3 = 4 - 3 = 1$ and $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} bx + 4 = 2b + 4$. For the limit to exist, the two one-sided limits must be equal, so $2b + 4 = 1$, or $b = -\frac{3}{2}$. This choice of b makes $\lim_{x \rightarrow 2} f(x)$ to be 1. Finally, for continuity we need $f(1) = \lim_{x \rightarrow 2} f(x)$, so $a = f(1) = 1$ for continuity.
10. By the picture for vector subtraction, the vector from A to B is $\vec{b} - \vec{a}$. The vector from A to D is thus $\frac{1}{3}(\vec{b} - \vec{a})$. Finally, by the picture for vector addition, the vector from O to D is the sum of the vector from O to A and the vector from A to D , i.e. $\vec{a} + \frac{1}{3}(\vec{b} - \vec{a}) = \frac{2}{3}\vec{a} + \frac{1}{3}\vec{b}$.