Answers to exam 1, version B

1. (a) See definition (2), p. 102 of our text.

   (b) Let $\varepsilon > 0$ be given. Set δ to be $\sqrt{\frac{\varepsilon}{2}}$. Then $0 < |x-2| < \sqrt{\frac{\varepsilon}{2}}$ implies $|x-2| < \sqrt{\frac{\varepsilon}{2}}$, thus $\left|(x-2)^2\right| < \frac{\varepsilon}{2}$, implying $\left|2x^2 - 8x + 8\right| < \varepsilon$. This is $\left|2x^2 - 8x - (-8)\right| < \varepsilon$, therefore $\lim_{x\to 2} 2x^2 - 8x = -8$.

2.

   (a)

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+1}{\sqrt{2x^2+3x}} &= \lim_{x\to\infty} \frac{(x+1)\,\frac{1}{x}}{\sqrt{2x^2+3x}\left(\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{(2x^2+3x)\left(\frac{1}{x^2}\right)}} \\
&= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{2+\frac{3}{x}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.
\end{aligned}$$

   (b)

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} &= \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right) \\
&= \lim_{x\to 3} \frac{x+1-4}{(x-3)\left(\sqrt{x+1}+2\right)} \\
&= \lim_{x\to 3} \frac{x-3}{(x-3)\left(\sqrt{x+1}+2\right)} \\
&= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}.
\end{aligned}$$

   (c) For $x < \frac{1}{2}$, $2x-1 < 0$, so $|2x-1| = -(2x-1)$ for those x's. Then

$$\begin{aligned}
\lim_{x\to \frac{1}{2}^-} \frac{|2x-1|}{2x^2+x-1} &= \lim_{x\to \frac{1}{2}^-} \frac{-(2x-1)}{2x^2+x-1} \\
&= \lim_{x\to \frac{1}{2}^-} \frac{-(2x-1)}{(2x-1)(x+1)} \\
&= \lim_{x\to \frac{1}{2}^-} \frac{-1}{x+1} = -\frac{2}{3}.
\end{aligned}$$

3. From the formula for component, we need

$$\frac{\langle 3,4\rangle \cdot \langle 1,x\rangle}{|\langle 3,4\rangle|} = 2,$$

   so

$$\frac{3+4x}{5} = 2,$$

   which gives $x = \dfrac{7}{4}$.

4. For x to be in the domain of g, we need $4 - 9x^2 \geq 0$, which is $x^2 \leq \frac{4}{9}$, whose solution is $-\dfrac{2}{3} \leq x \leq \dfrac{2}{3}$, so the domain is $\left[-\frac{2}{3}, \frac{2}{3}\right]$. For the range, notice that $4 - 9x^2$ must be a positive number less than or equal to 4, therefore we get numbers out of g from 0 to $2 = \sqrt{4}$. Therefore the range is $[0, 2]$.

5.

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\left(\vec{r}\left(1+h\right)-\vec{r}\left(1\right)\right) &= \lim_{h\to 0}\frac{1}{h}\left(\left\langle \left(1+h\right)^2+2\left(1+h\right),3\left(1+h\right)\right\rangle-\left\langle 3,3\right\rangle\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\left\langle h^2+4h,3h\right\rangle\right)\\ &= \lim_{h\to 0}\left\langle h+4,3\right\rangle=\left\langle 4,3\right\rangle.\end{aligned}$$

6. It's wrong to multiply the inequality by $x+1$ without considering the sign of $x+1$: if $x+1<0$, the direction of the inequality flips. Here's a correct way to do it: $\frac{x}{x+1}>2$ is true if $\frac{x}{x+1}-2>0$, which is the same as $\frac{x-2\left(x+1\right)}{x+1}>0$, i.e. $\frac{-x-2}{x+1}>0$, or $\frac{x+2}{x+1}<0$ (notice that the inequality flipped when I multiplied by -1). This is true only when $x+1$ and $x+2$ have different signs, which occurs only for $-2<x<-1$.

7. (a) No. The particle's x coordinate equals 4 when $t=2$, but at that time, the y coordinate is $6\cdot 2-4=8$, not 2.

   (b) The object's path is $x=t+2$, $y=6t-t^2$. Solve the first equation for t to get $t=x-2$. Plug that into the second equation to find that $y=6\left(x-2\right)-\left(x-2\right)^2$, which can be simplified to $y=-x^2+10x-16$.

8. Since $\sin^2\theta+\cos^2\theta=1$, we must have $\frac{1}{9}+\cos^2\theta=1$, so $\cos^2\theta=\frac{8}{9}$. Since $\theta\in\left(\frac{\pi}{2},\pi\right)$, $\cos\theta$ must be negative, therefore $\cos\theta=-\frac{\sqrt{8}}{3}$. Finally, $\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{1/3}{-\sqrt{8}/3}=-\frac{1}{\sqrt{8}}=-\frac{\sqrt{8}}{8}$.

9. We have $\lim_{x\to 2^-}f\left(x\right)=\lim_{x\to 2^-}2x-3=4-3=1$ and $\lim_{x\to 2^+}f\left(x\right)=\lim_{x\to 2^+}bx+4=2b+4$. For the limit to exist, the two one-sided limits must be equal, so $2b+4=1$, or $b=-\frac{3}{2}$. This choice of b makes $\lim_{x\to 2}f\left(x\right)$ to be 1. Finally, for continuity we need $f\left(1\right)=\lim_{x\to 2}f\left(x\right)$, so $a=f\left(1\right)=1$ for continuity.

10. By the picture for vector subtraction, the vector from A to B is $\vec{b}-\vec{a}$. The vector from A to C is thus $\frac{2}{3}\left(\vec{b}-\vec{a}\right)$. Finally, by the picture for vector addition, the vector from O to C is the sum of the vector from O to A and the vector from A to C, i.e. $\vec{a}+\frac{2}{3}\left(\vec{b}-\vec{a}\right)=\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}$.