Answers to exam 2, version A

1. (a) Using the chain rule twice,

$$\begin{aligned}\frac{dy}{dx} &= 2\sin\left(2x^3\right)\frac{d}{dx}\left(\sin\left(2x^3\right)\right)\\ &= 2\sin\left(2x^3\right)\cos\left(2x^3\right)\frac{d}{dx}\left(2x^3\right)\\ &= 12x^2\sin\left(2x^3\right)\cos\left(2x^3\right).\end{aligned}$$

(b) Use the quotient rule and then the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1\cdot\ln(2x+1) - x\frac{d}{dx}\left(\ln(2x+1)\right)}{\left(\ln(2x+1)\right)^2}\\ &= \frac{\ln(2x+1) - x\frac{2}{2x+1}}{\left(\ln(2x+1)\right)^2}.\end{aligned}$$

I'll leave it like this, although it can be simplified a little.

(c) Since $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{du}{dx}$,

$$\begin{aligned}\frac{d}{dx}\left(\arctan\left(\frac{1}{x}\right)\right) &= \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)\\ &= -\frac{1}{x^2+1}.\end{aligned}$$

2. Let a be the distance from the wall to the bottom of the ladder, and let b be the distance from the floor to the top of the ladder. From the Pythagorean theorem, $a^2+b^2=10^2$. Differentiate this relation with respect to time: $2a\frac{da}{dt}+2b\frac{db}{dt}=0$. Finally, plug in values. When $b=8$, $a=\sqrt{100-64}=\sqrt{36}=6$. We also know that $\frac{da}{dt}=2$, so

$$2\cdot 6\cdot 2 + 2\cdot 8\cdot\left(\frac{db}{dt}\right) = 0,$$

so $\frac{db}{dt} = -\frac{3}{2}$ feet per second.

3.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{4(x+h)}-\sqrt{4x}}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{4(x+h)}-\sqrt{4x}}{h}\left(\frac{\sqrt{4(x+h)}+\sqrt{4x}}{\sqrt{4(x+h)}+\sqrt{4x}}\right)\\ &= \lim_{h\to 0}\frac{4x+4h-4x}{h\left(\sqrt{4(x+h)}+\sqrt{4x}\right)}\\ &= \lim_{h\to 0}\frac{4}{\sqrt{4(x+h)}+\sqrt{4x}} = \frac{2}{\sqrt{4x}}.\end{aligned}$$

4. Using logarithmic differentiation, find $\frac{dy}{dx}$ if $y = \frac{x^{(x+1)}\left(x^3+2x+4\right)^5}{\sqrt{x^2+6}}$.

$$\ln y = (x+1)\ln x + 5\ln\left(x^3+2x+4\right) - \frac{1}{2}\ln\left(x^2+6\right),$$

so

$$\frac{1}{y}\frac{dy}{dx} = \ln x + (x+1)\frac{1}{x} + \frac{5\left(3x^2+2\right)}{x^3+2x+4} - \frac{1}{2}\left(\frac{2x}{x^2+6}\right),$$

thus

$$\frac{dy}{dx} = \frac{x^{(x+1)}\left(x^3+2x+4\right)^5}{\sqrt{x^2+6}}\left(\ln x + 1 + \frac{1}{x} + \frac{5\left(3x^2+2\right)}{x^3+2x+4} - \frac{x}{x^2+6}\right).$$

5.

(a) Using implicit differentiation, find $\frac{dy}{dx}$ if $x^2+xy+2y^2=7$. Differentiate the relationship with respect to x:

$$\frac{d}{dx}\left(x^2+xy+2y^2\right) = \frac{d}{dx}(7) = 0,$$

so

$$2x + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0,$$

where I had to use the product rule on xy. Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\left(\frac{2x+y}{x+4y}\right).$$

(b) From part a, $\frac{dy}{dx} = 0$ if $y = -2x$. We also need to be on the original curve, so plug $-2x$ in for y in the original equation: $x^2 + x(-2x) + 2(-2x)^2 = 7$, which is $7x^2 = 7$, or $x = 1, -1$. Since we need $y = -2x$, the points are $(1,-2)$ and $(-1,2)$.

6. The formula for Newton's method is that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

so in our case it's

$$x_{n+1} = x_n - \frac{x_n^4 + x_n - 1}{4x_n^3 + 1}.$$

Since $x_0 = 0$, $x_1 = 0 - \frac{0^4+0-1}{4\cdot 0^3+1} = 1$, and $x_2 = 1 - \frac{1^4+1-1}{4\cdot 1^3+1} = \frac{4}{5}$.

7. First we need a formula for population at any time t. Since the population is growing exponentially, it must satisfy $P(t) = P_0 e^{kt}$. Take t to be zero in 1990, so that $P(t) = 1500e^{kt}$. From the given data, setting t to be 10 we get

$$P(10) = 1800 = 1500e^{10k},$$

so

$$\frac{6}{5} = e^{10k},$$

and $\frac{1}{10}(\ln 6 - \ln 5) = k$. Therefore the population at any time t is $P(t) = 1500e^{\frac{t}{10}(\ln 6 - \ln 5)}$.

(a) Since this is 20 years after $t = 0$, the population will be $1500e^{\frac{20}{10}(\ln 6 - \ln 5)} = 1500e^{2(\ln 6 - \ln 5)}$.

(b) Solve $P(t) = 3000$ for t:

$$3000 = 1500e^{\frac{t}{10}(\ln 6 - \ln 5)},$$

so

$$2 = e^{\frac{t}{10}(\ln 6 - \ln 5)},$$

so that

$$t = \frac{10\ln 2}{\ln 6 - \ln 5}$$

years after 1990 (which turns out to be in 2028).

8. Since $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$, we have $f(4) = 2$, $f'(4) = \frac{1}{4}$, and $f''(4) = -\frac{1}{32}$. Plugging into the formulas for linear and quadratic approximation we get

$$L(x) = 2 + \frac{1}{4}(x-4),$$

for the linear approximation, and

$$P(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

for the quadratic approximation.

9. For this parameterized curve,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{6t^2-3}{2t}. \end{aligned}$$

This slope equals $\frac{3}{2}$ if $\frac{6t^2-3}{2t} = \frac{3}{2}$, so $12t^2 - 6 = 6t$, leading to the quadratic $12t^2 - 6t - 6 = 0$, i.e., $2t^2 - t - 1 = 0$. This factors as $(2t+1)(t-1) = 0$, so $t = 1, -\frac{1}{2}$ are the parameter values at which the slope is $\frac{3}{2}$. Plugging these in, the points are $(2, -1)$ and $\left(\frac{5}{4}, \frac{5}{4}\right)$.

10.

(a) By the chain rule, $A'(2) = f'(g(2))\,g'(2)$. Using the given values, $A'(2) = f'(1)\,g'(2) = -20$.

(b) $B'(1) = \frac{1}{2}(f(1) + 2g(1))^{-1/2}(f'(1) + 2g'(1)) = \frac{1}{2}(11)^{-1/2}(4-6) = -\frac{\sqrt{11}}{11}$.

(c) Since $g(2) = 1$, we must have $2 = g^{-1}(1)$. Therefore, $C'(1) = \frac{1}{g'(g^{-1}(1))} = \frac{1}{g'(2)} = -\frac{1}{5}$.