Answers to exam 2, version B

1. (a) Using the chain rule twice,

$$\begin{aligned}\frac{dy}{dx} &= 2\sin\left(4x^3\right)\frac{d}{dx}\left(\sin\left(4x^3\right)\right)\\ &= 2\sin\left(4x^3\right)\cos\left(4x^3\right)\frac{d}{dx}\left(4x^3\right)\\ &= 24x^3\sin\left(4x^3\right)\cos\left(4x^3\right).\end{aligned}$$

(b) Use the quotient rule and then the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1\cdot\ln(3x+1)-x\frac{d}{dx}(\ln(3x+1))}{(\ln(3x+1))^2}\\ &= \frac{\ln(3x+1)-x\frac{3}{3x+1}}{(\ln(3x+1))^2}.\end{aligned}$$

I'll leave it like this, although it can be simplified a little.

(c) Since $\frac{d}{dx}(\arctan u)=\frac{1}{1+u^2}\frac{du}{dx}$,

$$\begin{aligned}\frac{d}{dx}\left(\arctan\left(\frac{1}{x}\right)\right) &= \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)\\ &= -\frac{1}{x^2+1}.\end{aligned}$$

2. Let a be the distance from the wall to the bottom of the ladder, and let b be the distance from the floor to the top of the ladder. From the Pythagorean theorem, $a^2+b^2=10^2$. Differentiate this relation with respect to time: $2a\frac{da}{dt}+2b\frac{db}{dt}=0$. Finally, plug in values. When $b=8$, $a=\sqrt{100-64}=\sqrt{36}=6$. We also know that $\frac{da}{dt}=1$, so

$$2\cdot 6\cdot 1+2\cdot 8\cdot\left(\frac{db}{dt}\right)=0,$$

so $\frac{db}{dt}=-\frac{3}{4}$ feet per second.

3.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\left(\frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}}\right)\\ &= \lim_{h\to 0}\frac{2x+2h-2x}{h\left(\sqrt{2(x+h)}+\sqrt{2x}\right)}\\ &= \lim_{h\to 0}\frac{2}{\sqrt{2(x+h)}+\sqrt{2x}}=\frac{1}{\sqrt{2x}}.\end{aligned}$$

4.

$$\ln y=(x+1)\ln x+5\ln\left(x^3+3x+2\right)-\frac{1}{2}\ln\left(x^2+4\right),$$

so

$$\frac{1}{y}\frac{dy}{dx}=\ln x+(x+1)\frac{1}{x}+\frac{5\left(3x^2+3\right)}{x^3+3x+2}-\frac{1}{2}\left(\frac{2x}{x^2+4}\right),$$

thus

$$\frac{dy}{dx} = \frac{x^{(x+1)}\left(x^3+3x+2\right)^5}{\sqrt{x^2+4}}\left(\ln x + 1 + \frac{1}{x} + \frac{5\left(3x^2+3\right)}{x^3+3x+2} - \frac{x}{x^2+4}\right).$$

5. (a) Using implicit differentiation, find $\frac{dy}{dx}$ if $x^2+xy+2y^2=7$. Differentiate the relationship with respect to x:

$$\frac{d}{dx}\left(x^2+xy+2y^2\right) = \frac{d}{dx}(7) = 0,$$

so

$$2x+y+x\frac{dy}{dx}+4y\frac{dy}{dx}=0,$$

where I had to use the product rule on xy. Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\left(\frac{2x+y}{x+4y}\right).$$

(b) From part a, $\frac{dy}{dx}=0$ if $y=-2x$. We also need to be on the original curve, so plug $-2x$ in for y in the original equation: $x^2+x(-2x)+2(-2x)^2=7$, which is $7x^2=7$, or $x=1, -1$. Since we need $y=-2x$, the points are $(1,-2)$ and $(-1,2)$.

6. The formula for Newton's method is that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

so in our case it's

$$x_{n+1} = x_n - \frac{x_n^4+x_n-1}{4x_n^3+1}.$$

Since $x_0=0$, $x_1 = 0-\frac{0^4+0-1}{4\cdot 0^3+1}=1$, and $x_2 = 1-\frac{1^4+1-1}{4\cdot 1^3+1}=\frac{4}{5}$.

7. First we need a formula for population at any time t. Since the population is growing exponentially, it must satisfy $P(t)=P_0e^{kt}$. Take t to be zero in 1990, so that $P(t)=1200e^{kt}$. From the given data, setting t to be 10 we get

$$P(10) = 1500 = 1200e^{10k},$$

so

$$\frac{5}{4} = e^{10k},$$

and $\frac{1}{10}(\ln 5-\ln 4)=k$. Therefore the population at any time t is $P(t)=1200e^{\frac{t}{10}(\ln 5-\ln 4)}$.

(a) Since this is 20 years after $t=0$, the population will be $1200e^{\frac{20}{10}(\ln 5-\ln 4)}=1200e^{2(\ln 5-\ln 4)}$.

(b) Solve $P(t)=3000$ for t:

$$3000 = 1200e^{\frac{t}{10}(\ln 5-\ln 4)},$$

so

$$\frac{5}{2} = e^{\frac{t}{10}(\ln 5-\ln 4)},$$

so that

$$t = \frac{10(\ln 5-\ln 2)}{\ln 5-\ln 4}$$

years after 1990 (which turns out to be in 2031).

8. Since $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$, we have $f(4) = 2$, $f'(4) = \frac{1}{4}$, and $f''(4) = -\frac{1}{32}$. Plugging into the formulas for linear and quadratic approximation we get

$$L(x) = 2 + \frac{1}{4}(x-4),$$

for the linear approximation, and

$$P(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

for the quadratic approximation.

9. For this parameterized curve,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{6t^2-3}{2t}. \end{aligned}$$

This slope equals $\frac{3}{2}$ if $\frac{6t^2-3}{2t} = \frac{3}{2}$, so $12t^2 - 6 = 6t$, leading to the quadratic $12t^2 - 6t - 6 = 0$, i.e., $2t^2 - t - 1 = 0$. This factors as $(2t+1)(t-1) = 0$, so $t = 1, -\frac{1}{2}$ are the parameter values at which the slope is $\frac{3}{2}$. Plugging these in, the points are $(2, -1)$ and $\left(\frac{5}{4}, \frac{5}{4}\right)$.

10.

(a) By the chain rule, $A'(2) = f'(g(2))\,g'(2)$. Using the given values, $A'(2) = f'(1)\,g'(2) = -14$.

(b) $B'(1) = \frac{1}{2}(f(1) + 2g(1))^{-1/2}(f'(1) + 2g'(1)) = \frac{1}{2}(11)^{-1/2}(2-6) = -2\frac{\sqrt{11}}{11}$.

(c) Since $g(2) = 1$, we must have $2 = g^{-1}(1)$. Therefore, $C'(1) = \frac{1}{g'(g^{-1}(1))} = \frac{1}{g'(2)} = -\frac{1}{7}$.