Exam 2, version A
Math 171.501
Answers

An unsupported answer is a wrong answer!

1. (15 pts.) *You are given the values of f, f', g, and g' below:*

$$\begin{array}{llll} f(1)=2 & f'(1)=4 & g(1)=4 & g'(1)=-1 \\ f(2)=3 & f'(2)=7 & g(2)=3 & g'(2)=-5 \\ f(3)=5 & f'(3)=9 & g(3)=1 & g'(3)=-6 \end{array}$$

(a) *If $A(x)=f(x)\,g(x)$, determine $A'(1)$.* By the product rule $A'(x)=f'(x)\,g(x)+f(x)\,g'(x)$, so $A'(1)=f'(1)\,g(1)+f(1)\,g'(1)=4\cdot 4+2(-1)=14$.

(b) *If $B(x)=f(g(x))$, determine $B'(3)$.* From the chain rule, $B'(x)=f'(g(x))\,g'(x)$, so $B'(3)=f'(g(3))\,g'(3)=f'(1)\,g'(3)=4(-6)=-24$.

(c) *If $C(x)=\dfrac{1}{f(x)+g(x)}$, determine $C'(2)$.* This is done most easily by writing C as $(f(x)+g(x))^{-1}$, so $C'(x)=(-1)(f+g)^{-2}(f'+g')$, and $C'(2)=(-1)(f(2)+g(2))^{-2}(f'(2)+g'(2))=-\dfrac{7-5}{(3+3)^2}=-\dfrac{1}{18}$.

2. (10 pts.) *Find the derivative of $f(x)=\dfrac{x}{3x+1}$ using the definition, i.e., by evaluating a certain limit as h goes to zero. No credit for using another method (e.g., the quotient rule).* By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\frac{x+h}{3(x+h)+1}-\frac{x}{3x+1}}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{(3x+1)(x+h)-x(3x+3h+1)}{(3x+3h+1)(3x+1)}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{3x^2+x+3xh+h-3x^2-3xh-x}{(3x+3h+1)(3x+1)}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{(3x+3h+1)(3x+1)}\right) \\ &= \lim_{h\to 0}\frac{1}{(3x+3h+1)(3x+1)}=\frac{1}{(3x+1)^2}. \end{aligned}$$

3. (10 pts.) *At what point(s) on the parametric curve $x=2t^2+6t+1$, $y=4t^2+4t$ is the tangent line*

(a) *horizontal?* We seek points where $\dfrac{dy}{dx}=0$. Since $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{8t+4}{4t+6}$, this will be zero if $8t+4=0$, i.e., for $t=-\dfrac{1}{2}$. Plug this in to get the point as $\left(2\left(\dfrac{1}{4}\right)+6\left(-\dfrac{1}{2}\right)+1, 4\left(\dfrac{1}{4}\right)+4\left(-\dfrac{1}{2}\right)\right)=\left(-\dfrac{3}{2},-1\right)$.

(b) *parallel to the line $y=x$?* Since the line $y=x$ has slope 1, we look for points where $\dfrac{dy}{dx}=1$. Solving $\dfrac{8t+4}{4t+6}=1$, we get $8t+4=4t+6$, or $4t=2$, i.e., $t=\frac{1}{2}$. Therefore the point is $\left(2\left(\dfrac{1}{4}\right)+6\left(\dfrac{1}{2}\right)+1, 4\left(\dfrac{1}{4}\right)+4\left(\dfrac{1}{2}\right)\right)=\left(\dfrac{9}{2},3\right)$.

4. (10 pts.) *For the curve $x^3+x^2y-y^3=1$,*

(a) *use implicit differentiation to find dy/dx in terms of x and y.* Apply $\frac{d}{dx}$ to both sides:

$$3x^2 + 2xy + x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0,$$

where the middle two terms come from the product rule, and the last term has a factor of $\frac{dy}{dx}$ from the chain rule. Solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{3x^2 + 2xy}{3y^2 - x^2}.$$

(b) *find the equation of the tangent line at* $(1, 1)$. (Write this in slope-intercept form, i.e., $y = mx + b$ for the correct values of m and b.) The slope of the tangent is obtained by plugging $(1, 1)$ into the expression in part a, to get $\frac{5}{2}$. Thus the line is $y - 1 = \frac{5}{2}(x - 1)$. Since I asked for it in slope-intercept form, this is $y = \frac{5}{2}x - \frac{3}{2}$.

5. (10 pts.) *Find* $\frac{dy}{dx}$ *if* $y = \cos\left(e^{4x}\right)$. From the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= -\sin\left(e^{4x}\right) \frac{d}{dx}\left(e^{4x}\right) \\ &= -\sin\left(e^{4x}\right) e^{4x} \frac{d}{dx}(4x) \\ &= -4e^{4x} \sin\left(e^{4x}\right). \end{aligned}$$

6. (10 pts.) *Find* $\frac{d^2y}{dx^2}$ *if* $y = \tan x$. First, $\frac{dy}{dx} = \sec^2 x$. Take the derivative of this:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\sec^2 x\right) \\ &= 2\sec x \frac{d}{dx}(\sec x) \\ &= 2\sec^2 x \tan x. \end{aligned}$$

7. (10 pts.) *Suppose that we're using Newton's method to find the root of* $x^5 + x + 1 = 0$. *If* $x_0 = 0$, *what are* x_1 *and* x_2? The iteration formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^5 + x_n + 1}{5x_n^4 + 1}. \end{aligned}$$

If $x_0 = 0$, then $x_1 = 0 - \frac{0^5 + 0 + 1}{5 \cdot 0^4 + 1} = -1$, and then $x_2 = -1 - \frac{(-1)^5 + (-1) + 1}{5(-1)^4 + 1} = -1 + \frac{1}{6} = -\frac{5}{6}$.

8. (10 pts.) *Using differentials, approximate* $\sqrt{9.2}$. The differential approximation is that $f(x_0 + dx) \simeq f(x_0) + dx\, f'(x_0)$. Here we take $f(x) = x^{1/2}$, $x_0 = 9$, and $dx = 0.2$ to get

$$\begin{aligned} f(9 + 0.2) &\simeq f(9) + 0.2\, f'(9) \\ &\simeq 3 + 0.2 \cdot \frac{1}{2} 9^{-1/2} \\ &\simeq 3 + \frac{1}{30}. \end{aligned}$$

9. (15 pts.) *A water tank has the shape of a square pyramid with the vertex downward. The base (top) of the tank has a side of 5 feet, and the depth of the tank at the center is 10 feet. When the water is 6 ft deep at the center of the tank, it is observed that the water level is dropping at a rate of 1/5 ft/min due to a leak. How fast is the water leaking from the tank? Give units with your answer. (The volume of a pyramid is one third times the area of the base times the height.)* Take x to be the depth of water. The region the water occupies is a pyramid of height x feet and (by similar triangles) a base of side $\frac{x}{2}$. Thus the volume of water is $V = \frac{1}{3}\left(\frac{x}{2}\right)^2 x = \frac{x^3}{12}$. Differentiate this relationship implicitly with respect to time to get $\frac{dV}{dt} = \frac{x^2}{4}\frac{dx}{dt}$. When $x = 6$, we're told that $\frac{dx}{dt} = -\frac{1}{5}$. Plug that in to get $\frac{dV}{dt} = \frac{36}{4}\left(-\frac{1}{5}\right) = -\frac{9}{5}\,\text{ft}^3/\text{min}$.

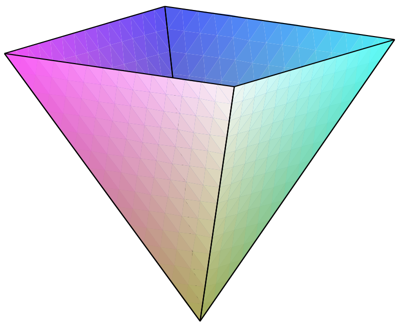

Figure for problem 9