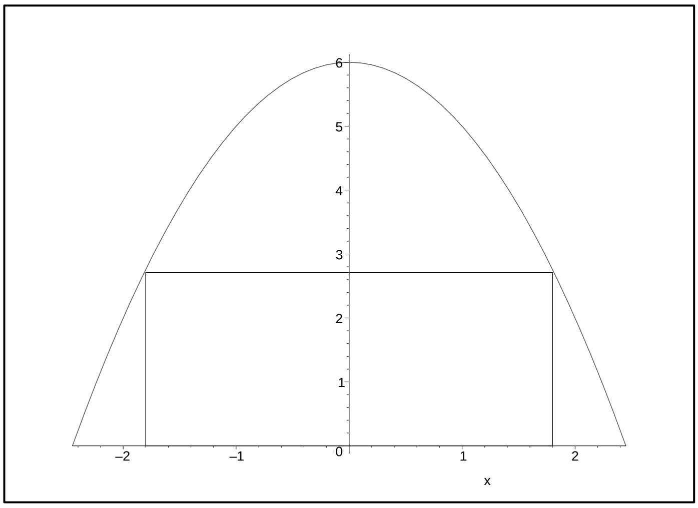
1. (5 pts.) *Give a precise statement of the hypotheses and conclusion of the Mean Value Theorem.* If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

2. (15 pts.) *Find the dimensions of the rectangle with largest area that has its base on the x axis and its other two vertices on the curve $y = 6 - x^2$ as shown below.*



Suppose that the upper right corner of the rectangle has coordinates (x, y). Then the base of the rectangle is $2x$, the height is y, so we want to maximize $2xy$, the area of the rectangle. Since the point is on the parabola, we must have $y = 6 - x^2$, so that's the constraint. Therefore, we want to maximize $A(x) = 2x(6 - x^2) = 12x - 2x^3$. The interval that we maximize on is $0 \le x \le \sqrt{6}$ (x can't be negative, because (x, y) is the upper right corner). Find critical points by setting $A'(x) = 12 - 6x^2$ to be zero. This is zero at $x = \pm\sqrt{2}$, but only $\sqrt{2}$ is in the interval. Now check values at the critical point and the endpoints: $A(0) = 0$, $A(\sqrt{2}) = 8\sqrt{2}$, and $A(\sqrt{6}) = 0$. Thus the maximum occurs when $x = \sqrt{2}$, givein a rectangle of base $2\sqrt{2}$ and height 4.

3. (5 pts.) *What are the domain and range of the function $f(x) = 4 \arccos(3x)$?* You can only take arccosine of numbers between -1 and 1, so $-1 \le 3x \le 1$, therefore the domain is $\left[-\frac{1}{3}, \frac{1}{3}\right]$. For the range, the angle that you get out of arccosine go from 0 to π. Since we're multiplying by 4, the range is $[0, 4\pi]$.

4. (10 pts.) *Find the function* $f(x)$ *which satisfies* $f'(x) = \sin x + \frac{1}{1+x^2}$, *and* $f(0) = 4$. Since an antiderivative of $\sin x$ is $-\cos x$, and an antiderivative of $\frac{1}{1+x^2}$ is $\arctan x$, the most general antiderivative of $\sin x + \frac{1}{1+x^2}$ is $-\cos x + \arctan x + C$, where C is a constant. Since we want $f(0)$ to be 4, we need $-\cos 0 + \arctan 0 + C = 4$. But $\cos 0 = 1$ and $\arctan 0 = 0$, so $C = 5$, and $f(x) = -\cos x + \arctan x + 5$.

5. (10 pts.) *For* $f(x) = 2x + \ln x$,

   (a) *find* $f^{-1}(2)$. *(Here,* f^{-1} *means inverse function, not reciprocal.)* Since $f(1) = 2$, we must have $1 = f^{-1}(2)$.

   (b) *find* $(f^{-1})'(2)$. The formula is that $(f^{-1})(x) = \frac{1}{f'(f^{-1}(x))}$. So, using part a,

   $$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3},$$

   using the fact that $f'(x) = 2 + \frac{1}{x}$, so $f'(1) = 3$.

6. (10 pts.) *Use logarithmic differentiation to find* $\frac{dy}{dx}$ *in terms of* x *if* $y = \frac{2^x x^{(x+3)}}{\sqrt{x^4+x^2+7}}$. Take logs of both sides and simplify using the laws of logarithms to get $\ln(y) = x\ln 2 + (x+3)\ln x - \frac{1}{2}\ln\left(x^4+x^2+7\right)$. Differentiate everything with respect to x to get

   $$\frac{1}{y}\frac{dy}{dx} = \ln 2 + \ln x + (x+3)\frac{1}{x} + \frac{1}{2}\frac{4x^3+2x}{x^4+x^2+7}.$$

   Multiply through by y and substitute back what y is in terms of x to get

   $$\frac{dy}{dx} = \frac{2^x x^{(x+3)}}{\sqrt{x^4+x^2+7}}\left(\ln 2 + \ln x + (x+3)\frac{1}{x} + \frac{1}{2}\frac{4x^3+2x}{x^4+x^2+7}\right).$$

7. (10 pts.) *The population of fire ants in my back yard grows exponentially. At* $t = 0$ *there are 200 fire ants, and* $t = 4$ *days there are 300 fire ants. When will there be 1000 fire ants?* Since the population grows exponentially, we have $P(t) = P_0 e^{kt}$. We're given that $P(0) = 200$, but that's P_0, so we know that $P(t) = 200e^{kt}$, for some value of k. Since $P(4) = 300$, we must have $200e^{4k} = 300$, so $e^{4k} = \frac{3}{2}$. Now take logs to see that $k = \frac{1}{4}\ln\left(\frac{3}{2}\right)$. So, we can write the population as $P(t) = 200e^{\frac{t}{4}\ln\left(\frac{3}{2}\right)}$. Now solve $P(t) = 1000$ for t: $1000 = 200e^{\frac{t}{4}\ln\left(\frac{3}{2}\right)}$, so $5 = e^{\frac{t}{4}\ln\left(\frac{3}{2}\right)}$. Take logs to solve for t to get $t = \frac{4\ln 5}{\ln 3 - \ln 2}$.

8. (10 pts.) *Determine* $\lim_{x\to 0} \frac{1-e^{3x^2}}{x^2}$. Since $e^0 = 1$ the limit is of the indeterminant form $\frac{0}{0}$, and we may use l'Hospital's rule. So,

$$\lim_{x\to 0} \frac{1-e^{3x^2}}{x^2} = \lim_{x\to 0} \frac{-6xe^{3x^2}}{2x} = \lim_{x\to 0} \frac{-6e^{3x^2}}{2} = -3.$$

The second equality is obtained by cancelling x; there's no reason to use l'Hospital's rule again.

9. (15 pts.) *Let* $f(x) = x + 2\cos x$ *on* $(0, 2\pi)$.

   (a) *On what open intervals is* f *increasing? On what open intervals is* f *decreasing?* Taking the derivative, $f'(x) = 1 - 2\sin x$. This is positive if $\sin x < \frac{1}{2}$ and is negative if $\sin x > \frac{1}{2}$. So, f' is positive on $(0, \frac{\pi}{6})$ and on $(\frac{5}{6}\pi, 2\pi)$, and f is increasing on those two intervals. We have $\sin x > \frac{1}{2}$ on $\left(\frac{\pi}{6}, \frac{5}{6}\pi\right)$, so f' is negative on that interval, and f is decreasing on that interval.

   (b) *Determine the* x *coordinates of any critical points of* f, *and classify them as local maxima, local minima, or neither.* Critical points are where either f' equals zero or is undefined. Since f' is defined everywhere, the only type of critical point is where the derivative is zero, which occurs at $x = \frac{\pi}{6}$ and $x = \frac{5}{6}\pi$. Looking at the answer to part a and using the first derivative test, f has a local maximum at $x = \frac{\pi}{6}$ and a local minimum at $x = \frac{5}{6}\pi$.

   (c) *On what open intervals is* f *concave up? On what open intervals is* f *concave down?* Take the derivative of f' to get $f'' = -2\cos x$. This positive where $\cos x < 0$, i.e., on $\left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$, so that f is concave up on this interval, and f'' is negative where $\cos x$ is positive, i.e., on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3}{2}\pi, 2\pi\right)$, so f is concave down on these two intervals.

10. (10 pts.) *For what* x*'s, if any, does* $\ln(x) + \ln(2x+3) = \ln 2$*?* Using laws of logarithms, $\ln(x) + \ln(2x+3) = \ln(x(2x+3)$, so we want $\ln(x(2x+3)) = \ln 2$. This is true if $x(2x+3) = 2$, or $2x^2 + 3x - 2 = 0$. This factors to $(2x-1)(x+2) = 0$, so the roots are $x = \frac{1}{2}$ and $x = -2$. However, $x = -2$ doesn't satisfy the original equation, since $\ln(-2)$ is undefined. Thus the only solution is $x = \frac{1}{2}$.