**Theorem:** Continuous images of connected sets are connected, i.e., if $E \subseteq \mathbb{R}^n$ with $E$ connected, and if $f : E \rightarrow \mathbb{R}^m$ is continuous, then $f(E)$ is connected.

**Proof:** I’ll prove the contrapositive, i.e., if $f(E)$ is not connected, then $E$ is not connected. So, suppose that $f(E)$ is not connected. Then there exist open sets $A, B$ in $\mathbb{R}^m$ with $A \cap f(E) \neq \emptyset$, $B \cap f(E) \neq \emptyset$, $A \cap B = \emptyset$, and $f(E) \subseteq A \cup B$. Look at $f^{-1}(A)$, $f^{-1}(B)$. I claim that these separate $E$, using the definition of separation involving relatively open sets. Well, $f^{-1}(A) \neq \emptyset$ since there is a $y \in A \cap f(E)$ and similarly $f^{-1}(B) \neq \emptyset$. Can there be any $x \in f^{-1}(A) \cap f^{-1}(B)$? If so, then $f(x) \in A$ and $f(x) \in B$, but $A \cap B = \emptyset$, so we know that this can’t occur. Finally, $E = f^{-1}(A) \cup f^{-1}(B)$ since $x \in E$ implies $f(x) \in f(E)$, implying $f(x) \in A$ or $f(x) \in B$, hence $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. Thus $E$ is not connected.

Here is an example of applying the machinery we’ve created.

**Extreme Value Theorem:** Suppose that $H \subseteq \mathbb{R}^n$ is compact, and $f : H \rightarrow \mathbb{R}$ is continuous. Then $M = \sup_{x \in H} f(x)$ and $m = \inf_{x \in H} f(x)$ are finite numbers. Moreover, there exist $x_M$ and $x_m$ in $H$ so that $f(x_M) = M$ and $f(x_m) = m$.

**Proof:** $f(H)$ is compact, hence closed and bounded. Hence $M$ and $m$ are finite (since $f(H)$ is bounded), and in fact elements of $f(H)$ (since $f(H)$ is closed).
Partial derivatives and partial integrals (11.1)

Essentially treating all but one variable in a function on $\mathbb{R}^n$ as a constant, and then differentiating or integrating with respect to the remaining variable. In other words, if $f$ is $f(x_1, \cdots, x_n)$, then to define $\int_a^b f(x_1, \cdots, x_n) \, dx_j$, let

$$g(t) = f(x_1, \cdots, x_{j-1}, t, x_{j+1}, \cdots, x_n),$$

where all the rest of the $x_i$'s are being held constant, and then $\int_a^b f(x_1, \cdots, x_n) \, dx_j$ is defined to be $\int_a^b g(t) \, dt$. Similarly, define $\frac{\partial f}{\partial x_j}$ at $(x_1, \cdots, x_n)$ to be $g'(x_j)$.

**Part 1: partial derivatives**

In terms of the limit definition of derivative,

$$\frac{\partial f}{\partial x_j}(x_1, \cdots, x_n) = \lim_{h \to 0} \frac{f(x_1, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}.$$ 

Other notation for the partial derivative is $f_{x_j}$. Higher order partials:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_j} \right)$$

etc. Note that

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} (f_x) = \frac{\partial^2 f}{\partial y \partial x}$$

so that the order reverses (ouch!). The *order* of a partial is the number of partials you’re taking: $f_{xxyy}$ is fourth order, for example. Higher order partials are *pure* if you’re taking all of the partials with respect to the same variable, otherwise they are *mixed*. One of the main results of this section is that, under appropriate hypotheses, it doesn’t matter what order you take mixed partials in (so that $f_{xy} = f_{yx}$ under appropriate hypotheses).
What about partials of vector-valued functions? Well, if $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, then

$$\frac{\partial}{\partial x_i} (f) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \begin{pmatrix} f_1(x_1, \ldots, x_i + h, \ldots, x_n) - f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_i + h, \ldots, x_n) - f_m(x_1, \ldots, x_n) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x_i} (f_1) \\ \vdots \\ \frac{\partial}{\partial x_i} (f_m) \end{pmatrix},$$

in other words, you just take the partial derivatives of each component function.

**Definition:** If $V$ is open in $\mathbb{R}^n$, $f : V \to \mathbb{R}^m$, $p \in \mathbb{N}$. Then $f \in C^p (V)$ iff all partial derivatives of all orders $k \leq p$ exist and are continuous on $V$. $f \in C^\infty (V)$ if it is in $C^p (V)$ for all $p$, i.e., $f$ has continuous partials of all order.

**Note:** Usual differentiation rules apply to partials. For example, $\frac{\partial}{\partial x} (fg) = f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x}$. Reason is that you simply treat the other variables as constant, and it’s a regular one-dimensional derivative. We also have the Mean Value Theorem:

**Theorem:** If $\forall y \in [c, d]$ we have that $f(x, y)$ is continuous as a function of $x$ on the interval $[a, b]$ and differentiable as a function of $x$ on $(a, b)$, then $\forall y \in [c, d]$, $\exists c \in (a, b)$ so that

$$\frac{f(b, y) - f(a, y)}{b - a} = \frac{\partial f}{\partial x} (c, y).$$

In general $c$ will depend on $y$. Another way of writing the MVT (which we’ll need momentarily) is the following: instead of $a$ and $b$, consider the interval for $x$ to be from $x_0$ to $x_0 + h$. We can then say

$$\frac{f(x_0 + h, y) - f(x_0, y)}{h} = f_x (x_0 + \theta h, y),$$

for some $\theta \in (0, 1)$.

**Clairaut’s Theorem (equality of mixed partials):** (I’ll state this in $\mathbb{R}^2$ for convenience, and then make some comments about $\mathbb{R}^n$.) Suppose that $V$ is open in $\mathbb{R}^2$, $(a, b) \in V$, $f : V \to \mathbb{R}$. Suppose that $f \in C^1 (V)$, that one of the 2nd order partials is defined on $V$ and is continuous at $(a, b)$, say $f_{yx}$. Then the other second order mixed partial exists at $(a, b)$, and we have $f_{yx} (a, b) = f_{xy} (a, b)$.

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