Problem 1: We will want to find a and b so that $E(a, b) = \sum_{i=1}^{n}\left(y_i - \left(ax_i + \frac{b}{x_i}\right)\right)^2$ is minimized. Set the partials to be zero:

$$E_a = 2\sum_{i=1}^{n}\left(y_i - ax_i - \frac{b}{x_i}\right)(-x_i) = 0$$

$$E_b = 2\sum_{i=1}^{n}\left(y_i - ax_i - \frac{b}{x_i}\right)\left(\frac{-1}{x_i}\right) = 0.$$

Collecting terms and recalling that $\sum_{i=1}^{n} 1 = n$, this gives two equations in two unknowns:

$$a\sum_{i=1}^{n} x_i^2 + bn = \sum_{i=1}^{n} x_i y_i$$

$$an + b\sum_{i=1}^{n} x_i^{-2} = \sum_{i=1}^{n} x_i^{-1} y_i.$$

From Cramer's rule, we get

$$a = \frac{det\begin{bmatrix} \sum_{i=1}^{n} x_i y_i & n \\ \sum_{i=1}^{n} x_i^{-1} y_i. & \sum_{i=1}^{n} x_i^{-2} \end{bmatrix}}{det\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & n \\ n & \sum_{i=1}^{n} x_i^{-2} \end{bmatrix}}$$

and

$$b = \frac{det\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i \\ n & \sum_{i=1}^{n} x_i^{-1} y_i. \end{bmatrix}}{det\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & n \\ n & \sum_{i=1}^{n} x_i^{-2} \end{bmatrix}}.$$

Problem 2:

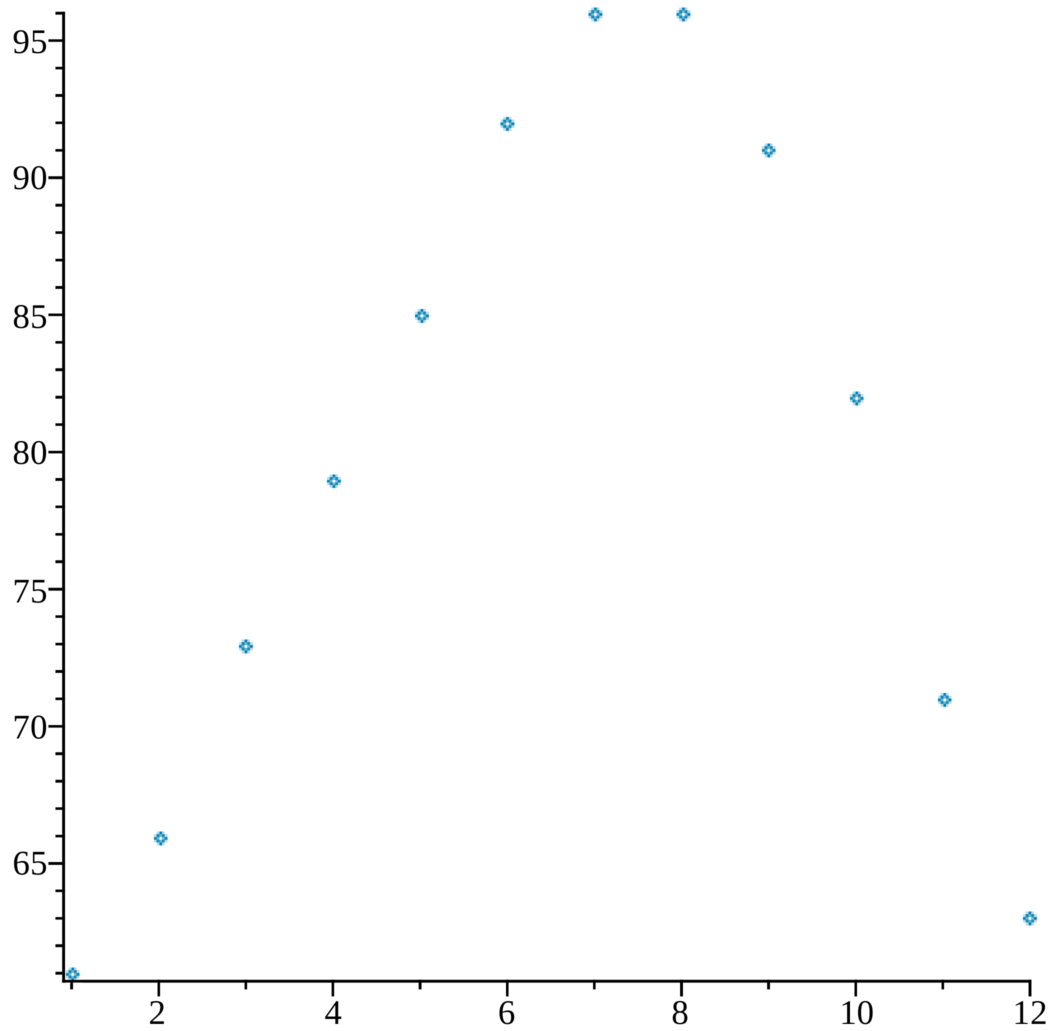
$with(Statistics):$
$with(plots):$
I'll call the month array M and the temperature array T.
$T := [61, 66, 73, 79, 85, 92, 96, 96, 91, 82, 71, 63];$

$$[61, 66, 73, 79, 85, 92, 96, 96, 91, 82, 71, 63] \quad \textbf{(1)}$$

$M := [seq(i, i = 1..12)];$

$$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] \quad \textbf{(2)}$$

$ScatterPlot(M, T);$



The above is the scatter plot of temperature.
I'll need to find a, b, and c to minimize E, the sum of the squares of the errors. I'll supress output so that you aren't printing out too much garbage.

$$E := \sum_{i=1}^{12} \left(T_i - \left(a + b \cdot \sin\left(\frac{\pi}{6} \cdot M_i + c \right) \right) \right)^2 :$$

$eq1 := diff(E, a) = 0:$
$eq2 := diff(E, b) = 0:$
$eq3 := diff(E, c) = 0:$
From the scatter plot, we expect a sine curve going from about 60 to about 96, which would have an amplitude of 18. So, I'll tell it to look for b between 10 and 20. I'm taking c to be between 1 and 12 since

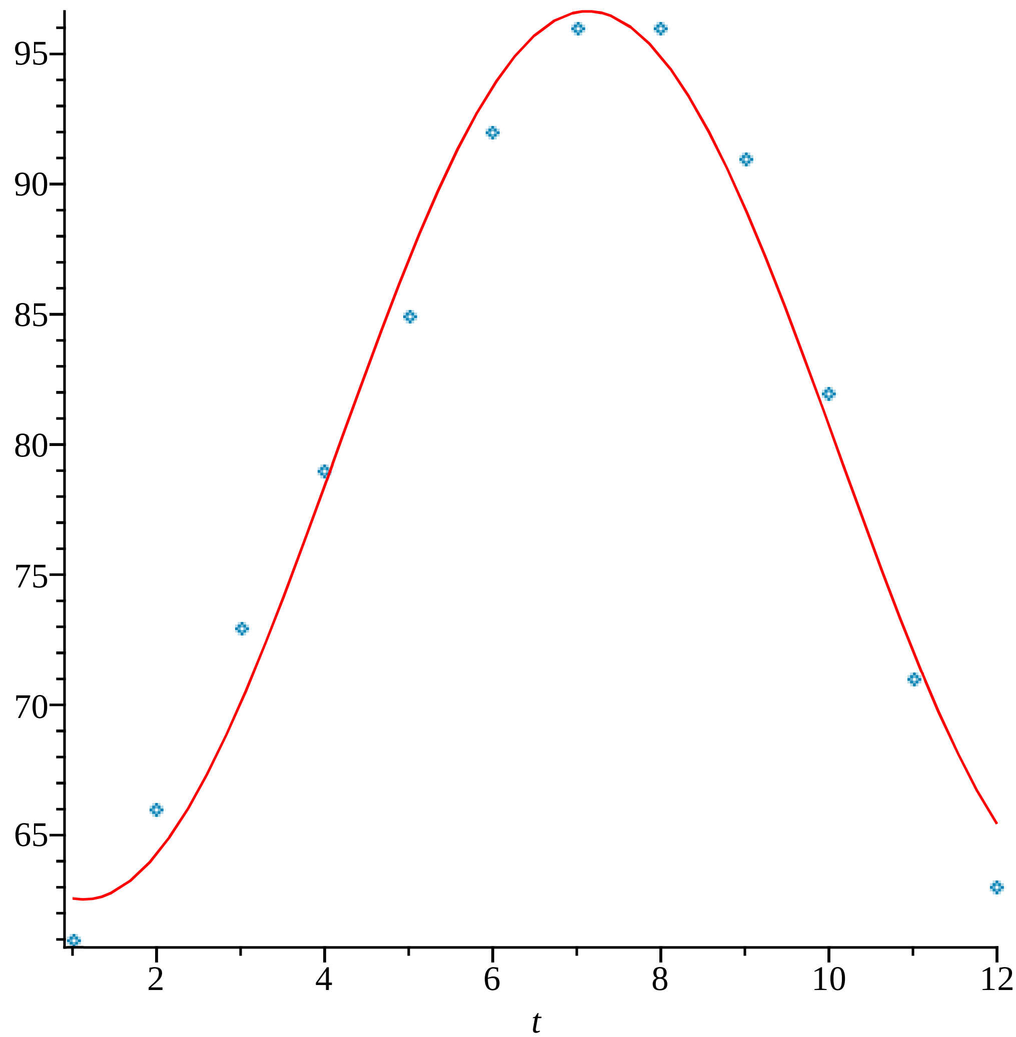
the function is periodic. On the other hand, c's outside of this range aren't wrong.
$sol := fsolve(\{eq1, eq2, eq3\}, \{a, b, c\}, \{b = 10..20, c = 1..12\});$

$$\{a = 79.58333333, b = 17.05545660, c = 4.120684061\} \quad (3)$$

So, here's the answer.
$subs\left(sol, a + b \cdot sin\left(\frac{\pi}{6} \cdot t + c\right)\right);$

$$79.58333333 + 17.05545660 \sin\left(\frac{1}{6} \pi t + 4.120684061\right) \quad (4)$$

Let's plot this over the scatter plot to see how close we are.
$p1 := ScatterPlot(M, T):$
$p2 := plot\left(subs\left(sol, a + b \cdot sin\left(\frac{\pi}{6} \cdot t + c\right)\right), t = 1..12\right):$
$display([p1, p2]);$



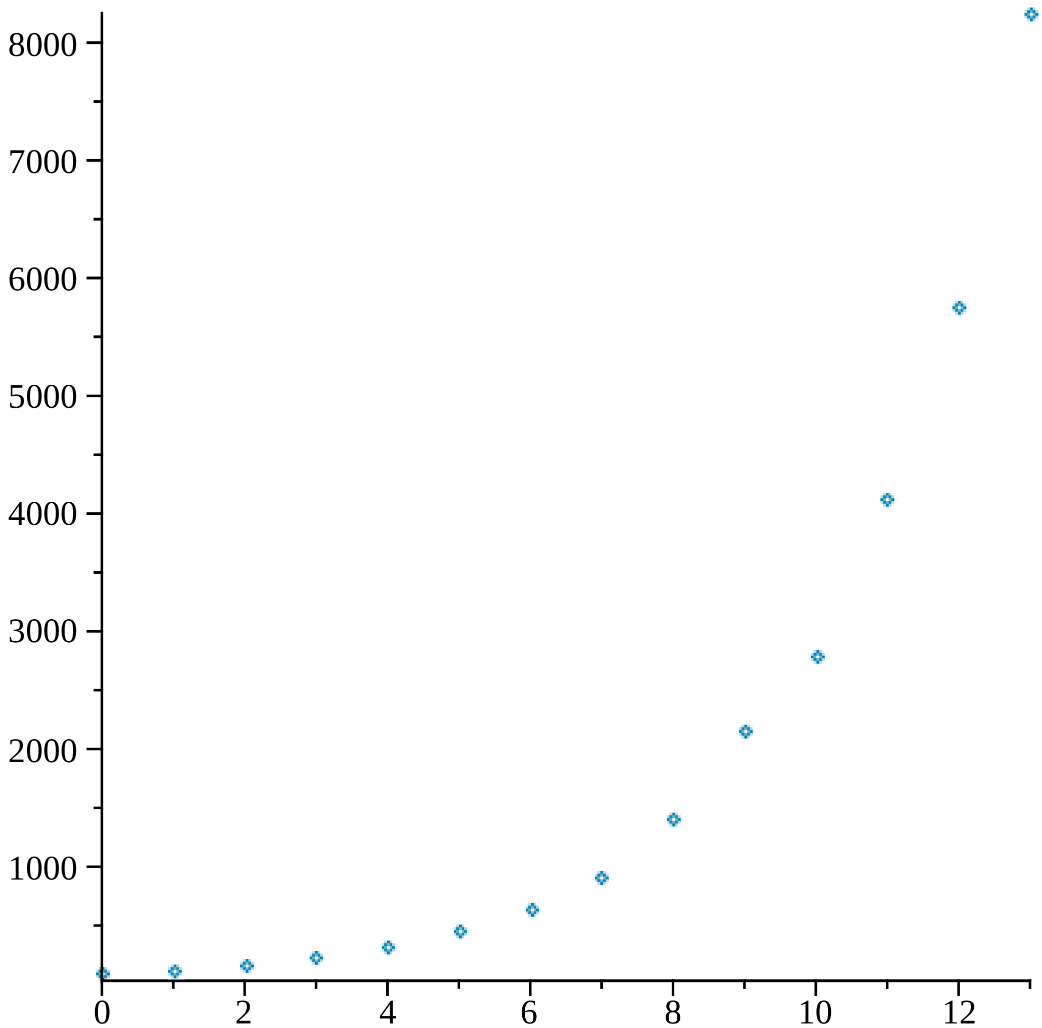
Not too bad. If you don't restrict b to keep it away from zero, you get another solution to the equations which has b identically zero. This is a critical point but not a minimum; it's a saddle point. One moral is: don't assume that a critical point which you find is necessarily the answer which you are looking for.
Problem 3. I'll define *Day* to be the array of days, *Frond* to be the array of frond numbers.
$Day := [seq(i, i = 0..13)];$

$$[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] \quad \textbf{(5)}$$

Frond := [100, 127, 171, 233, 323, 452, 654, 918, 1406, 2150, 2800, 4140, 5760, 8250];

$$[100, 127, 171, 233, 323, 452, 654, 918, 1406, 2150, 2800, 4140, 5760, 8250] \quad \textbf{(6)}$$

ScatterPlot(*Day*, *Frond*);

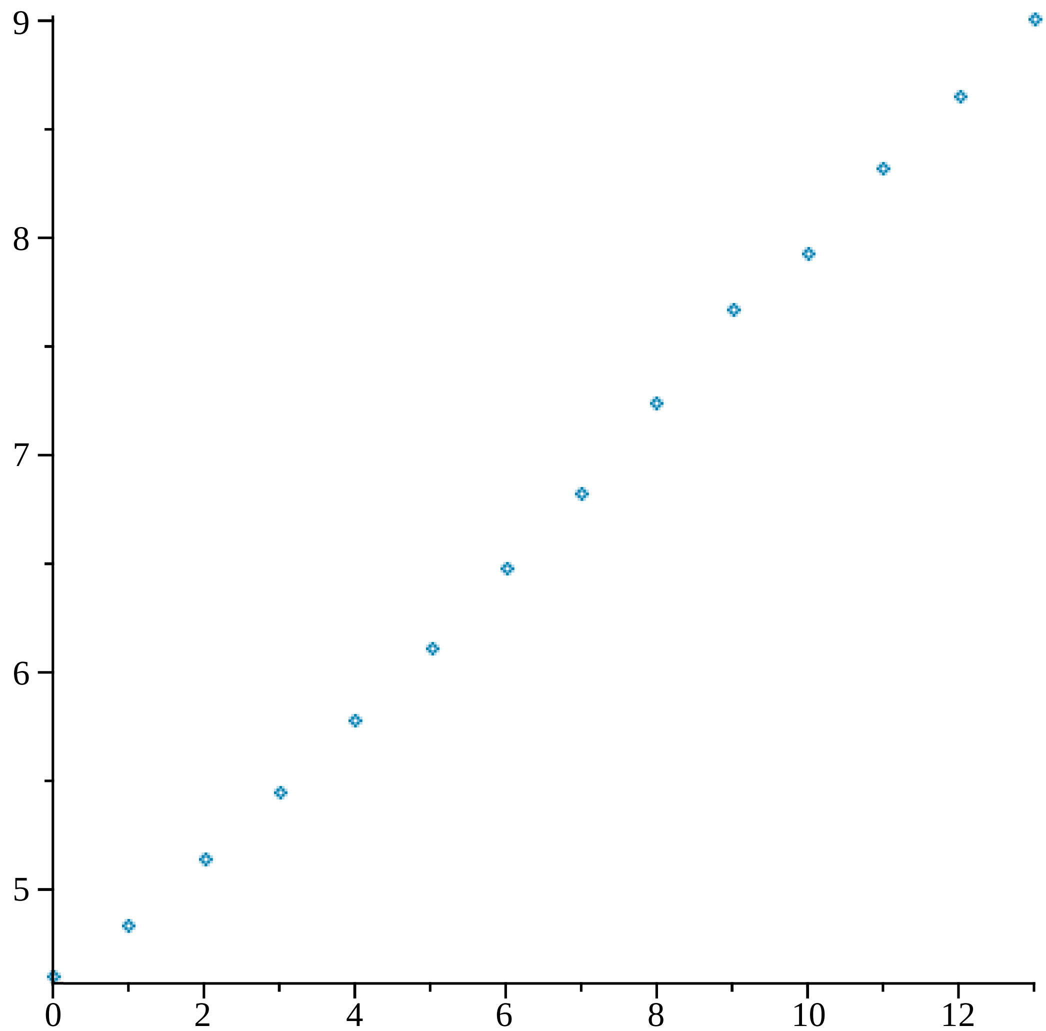


This looks remarkably exponential. Let's plot the log of the number of fronds.

lFrond := *map*(ln, *Frond*);

$$[2\ln(10), \ln(127), \ln(171), \ln(233), \ln(323), \ln(452), \ln(654), \ln(918), \ln(1406), \ln(2150), \ln(2800), \ln(4140), \ln(5760), \ln(8250)] \quad \textbf{(7)}$$

ScatterPlot(*Day*, *lFrond*);



(8)

This looks close to linear. I'll find the least squares line using the built-in function in the CurveFitting package. *with*(*CurveFitting*);

[*ArrayInterpolation, BSpline, BSplineCurve, Interactive, LeastSquares, PolynomialInterpolation, RationalInterpolation, Spline, ThieleInterpolation*] **(9)**

LeastSquares(*Day*, *lFrond*, *t*);

$$\left(\frac{11}{455}\ln(5760)-\frac{2}{35}\ln(10)+\frac{1}{35}\ln(8250)-\frac{11}{455}\ln(127)-\frac{1}{455}\ln(654)+\frac{1}{455}\ln(918)-\frac{9}{455}\ln(171)+\frac{3}{455}\ln(1406)+\frac{1}{91}\ln(2150)-\frac{1}{65}\ln(233)+\frac{9}{455}\ln(4140)-\frac{1}{91}\ln(323)-\frac{3}{455}\ln(452)+\frac{1}{65}\ln(2800)\right)t+\frac{18}{35}\ln(10)-\frac{4}{35}\ln(8250)+\frac{8}{35}\ln(127)+\frac{3}{35}\ln(654)+\frac{2}{35}\ln(918)+\frac{1}{5}\ln(171)+\frac{1}{35}\ln(1406)+\frac{6}{35}\ln(233)-\frac{2}{35}\ln(4140)+\frac{1}{7}\ln(323)+\frac{4}{35}\ln(452)-\frac{3}{35}\ln(5760)-\frac{1}{35}\ln(2800) \quad \textbf{(10)}$$

sol1 := *evalf*(%);

$$0.3486434968\, t + 4.455453892 \quad \textbf{(11)}$$

This is saying that the log of the number of fronds is close to the above line. This is predicting that the number of fronds is *e* raised to the above linear function.

exp(*sol1*); *expand*(%);

$$e^{0.3486434968\, t + 4.455453892}$$

$$86.09521991\, e^{0.3486434968\, t} \quad \textbf{(12)}$$

Notice that I used "expand" to get the constant times the exponential. We'll need that in finding the actual least squares fit. It's $e^{4.455453893}$. You can find this other ways as well.

Anyway, this is saying that the number of fronds is approximately the above exponential. Now let's do a least squares fit of an exponential directly to the data. This won't give us exactly the same answer, since we aren't minimizing quite the same thing.

$$E := \sum_{i=1}^{14} \left(Frond_i - A \cdot exp\left(Day_i \cdot r \right) \right)^2;$$

$$(100 - A)^2 + \left(127 - A\, e^{r}\right)^2 + \left(171 - A\, e^{2r}\right)^2 + \left(233 - A\, e^{3r}\right)^2 + \left(323 - A\, e^{4r}\right)^2 + \left(452 - A\, e^{5r}\right)^2 + \left(654 - A\, e^{6r}\right)^2 + \left(918 - A\, e^{7r}\right)^2 + \left(1406 - A\, e^{8r}\right)^2 + \left(2150 - A\, e^{9r}\right)^2 + \left(2800 - A\, e^{10r}\right)^2 + \left(4140 - A\, e^{11r}\right)^2 + \left(5760 - A\, e^{12r}\right)^2 + \left(8250 - A\, e^{13r}\right)^2 \quad \textbf{(13)}$$

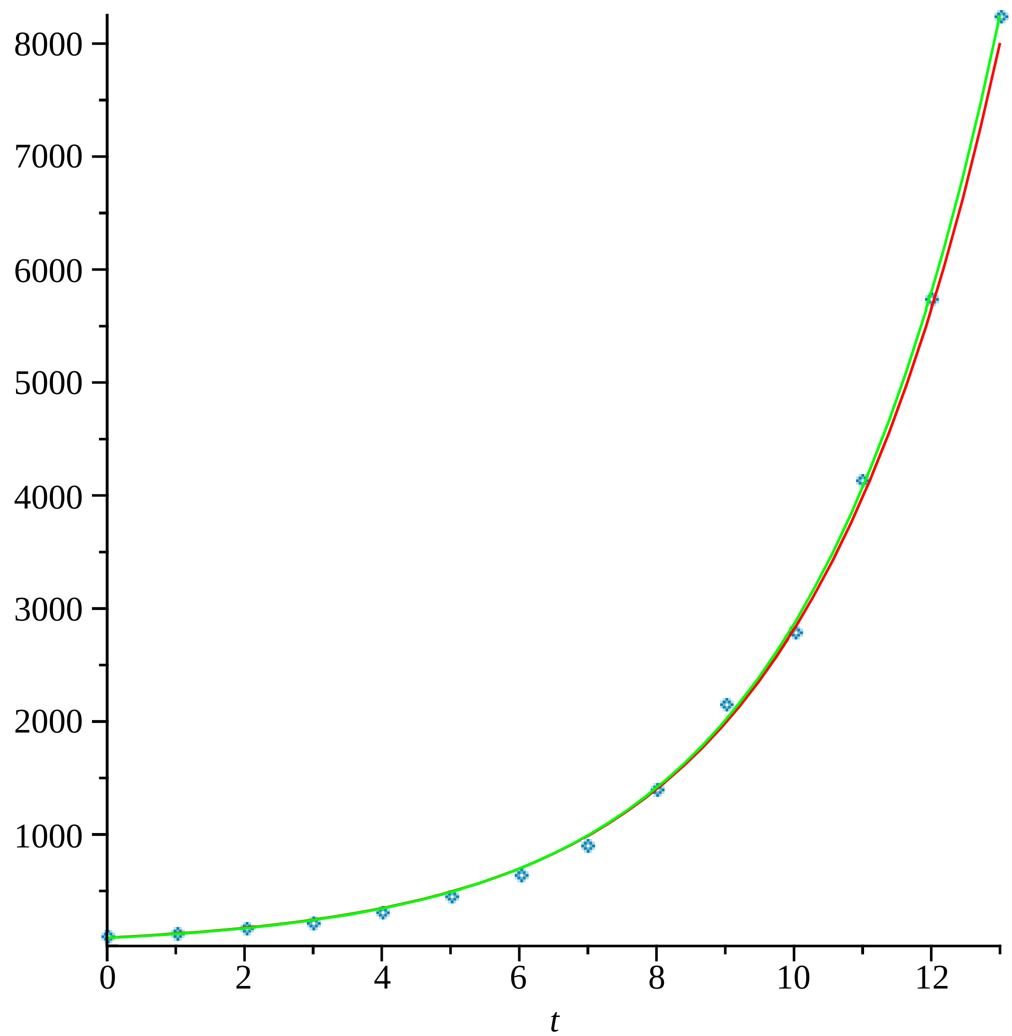
eq1 := *diff*(*E*, *A*) = 0 : *eq2* := *diff*(*E*, *r*) = 0 :

Maple chokes if we don't give it reasonable intervals for fsolve. But from the linear fit to the log of the data, we have a pretty good idea of what the answer should be.

sol2 := *fsolve*({*eq1*, *eq2*}, {*A*, *r*}, {*A* = 80 ..90, *r* = 0.3 ..0.4});

$$\{A = 83.89416369,\ r = 0.3529719347\} \quad \textbf{(14)}$$

p1 := *ScatterPlot*(*Day*, *Frond*) :
p2 := *plot*(*exp*(*sol1*), *t* = 0 ..13, *color* = *red*) :
p3 := *plot*(*subs*(*sol2*, *A*· *exp*(*r*· *t*)), *t* = 0 ..13, *color* = *green*) :
display([*p1*, *p2*, *p3*]);



Obviously these are very close. The linear fit using logarithms is the one which is slightly smaller for large time. Please be aware that I'm not claiming that these are two different methods for fitting the data. The second exponential which we found is by definition the best least squares fit. The point of doing the least squares fit to the logarithm of the data was to give a preliminary estimate of what the exponential should be, so that we could put reasonable intervals into fsolve.

Problem 4 *restart* :

Let's enter the data into arrays: hw1, hw2, and final.

> $hw1 := [13.5, 13, 14.5, 13, 18.5, 19.5, 16.5, 12, 18.5, 16];$

$$hw1 := [13.5, 13, 14.5, 13, 18.5, 19.5, 16.5, 12, 18.5, 16] \quad \textbf{(15)}$$

> $hw2 := [17.75, 8, 15.25, 14.5, 17.25, 14.5, 12.75, 15.25, 15.75, 15.75];$

$$hw2 := [17.75, 8, 15.25, 14.5, 17.25, 14.5, 12.75, 15.25, 15.75, 15.75] \quad \textbf{(16)}$$

> $final := [80.6, 66.3, 54.3, 76.5, 86.0, 77.6, 84.1, 81.4, 81.9, 91.2];$

$$final := [80.6, 66.3, 54.3, 76.5, 86.0, 77.6, 84.1, 81.4, 81.9, 91.2] \quad \textbf{(17)}$$

We minimize the sum of the squares, as usual. Find the critical point by setting the partials to be zero.

> $E := \sum_{i=1}^{10} \left(final_i - \left(a + b \cdot hw1_i + c \cdot hw2_i\right)\right)^2 :$

> $eq1 := diff(E, a) = 0 :$

> $eq2 := diff(E, b) = 0 :$

> $eq3 := diff(E, c) = 0:$

We can use "solve" rather than "fsolve" since the system must be linear, and it's easy to solve linear systems exactly. The *a*, *b*, *c* below give the best fit.

> $sol := solve(\{eq1, eq2, eq3\}, \{a, b, c\});$

$$sol := \{a = 44.09098338, b = 0.9951376013, c = 1.258901792\} \quad \textbf{(18)}$$

> $subs(sol, a + b \cdot 17 + c \cdot 16.75);$

$$82.09492762 \quad \textbf{(19)}$$

So, a student who got a 17 and a 16.75 is predicted to end up with a final grade of 82. This all looks pretty bogus, however, since the formula only predicts an 89 for a student who got 20's on both assignments.