

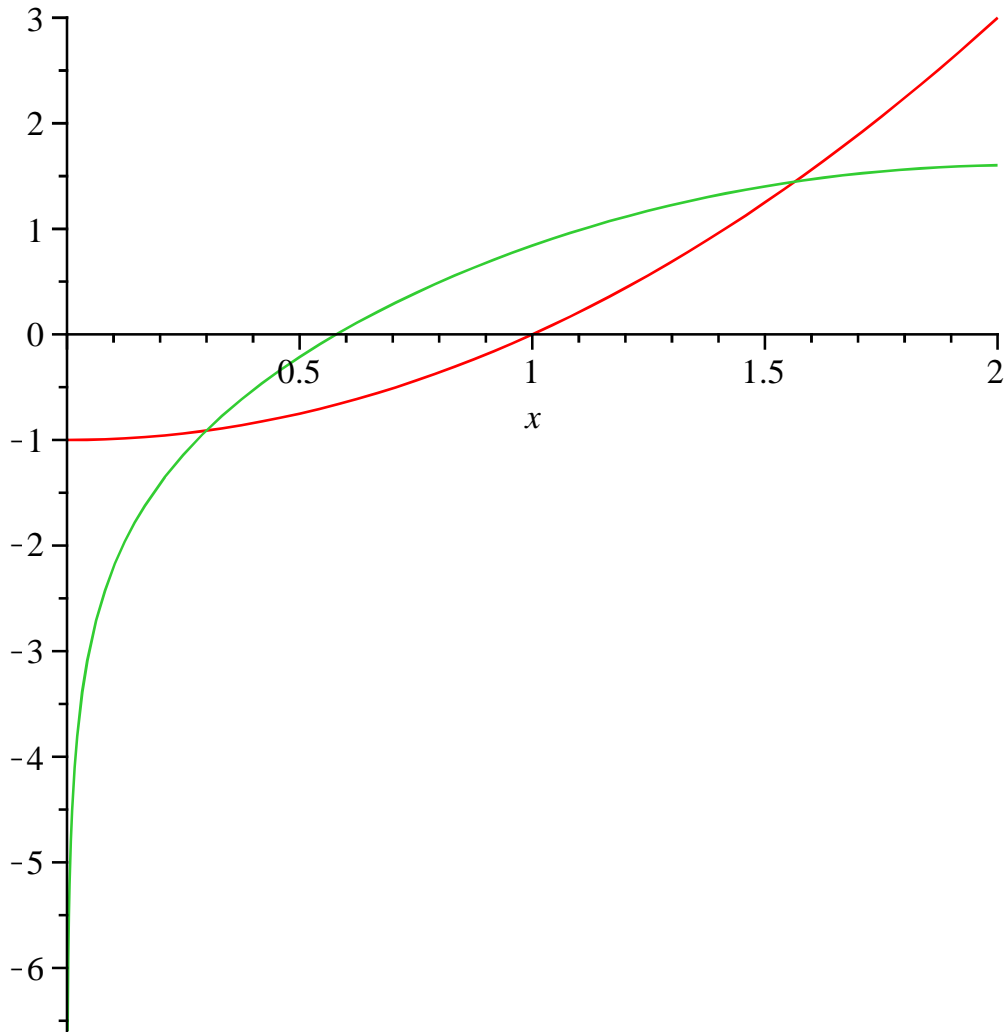
Maple problems:

1) Here's a plot of $y = \ln(x) + \sin(x)$ and $y = x^2 - 1$ on the same set of axes. I'm defining them to be functions so that I don't have to type them in several times. You can also define them to be expressions. A little experimentation shows that there are only two intersections.

```
> f := x -> ln(x) + sin(x); g := x -> x^2 - 1;
      f := x -> ln(x) + sin(x)
      g := x -> x^2 - 1
```

(1)

```
> plot( {f(x), g(x)}, x=0..2);
```



```
> a := fsolve(f(x) = g(x), x=0..0.5);
      a := 0.2995808181
```

(2)

```
> b := fsolve(f(x) = g(x), x=1..2);
      b := 1.564454181
```

(3)

```
> Int(f(x) - g(x), x=a..b); evalf(%);
      ∫0.29958081811.564454181 (ln(x) + sin(x) - x^2 + 1) dx
      0.7429950034
```

(4)

The area between the curves is therefore 0.7429950034.

Problem 2) Define $p(x)$ to be $ax^3 + bx^2 + cx + d$. We need $p(-2)$ to be -1 and $p(4)$ to be 3 . We also need derivatives of p at -2 and 4 to be zero, and we'll see if this gives us these as local minimum and local maximum. I'm using restart to clear the values of the variables (especially a and b).

$$\begin{aligned} > \text{restart;} \\ > p := x \rightarrow a \cdot x^3 + b \cdot x^2 + c \cdot x + d; \\ & \qquad \qquad \qquad p := x \rightarrow a x^3 + b x^2 + c x + d \end{aligned} \tag{5}$$

$$\begin{aligned} > eq1 := p(-2) = -1; \\ & \qquad \qquad \qquad eq1 := -8 a + 4 b - 2 c + d = -1 \end{aligned} \tag{6}$$

$$\begin{aligned} > eq2 := D(p)(-2) = 0; \\ & \qquad \qquad \qquad eq2 := 12 a - 4 b + c = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} > eq3 := p(4) = 3; \\ & \qquad \qquad \qquad eq3 := 64 a + 16 b + 4 c + d = 3 \end{aligned} \tag{8}$$

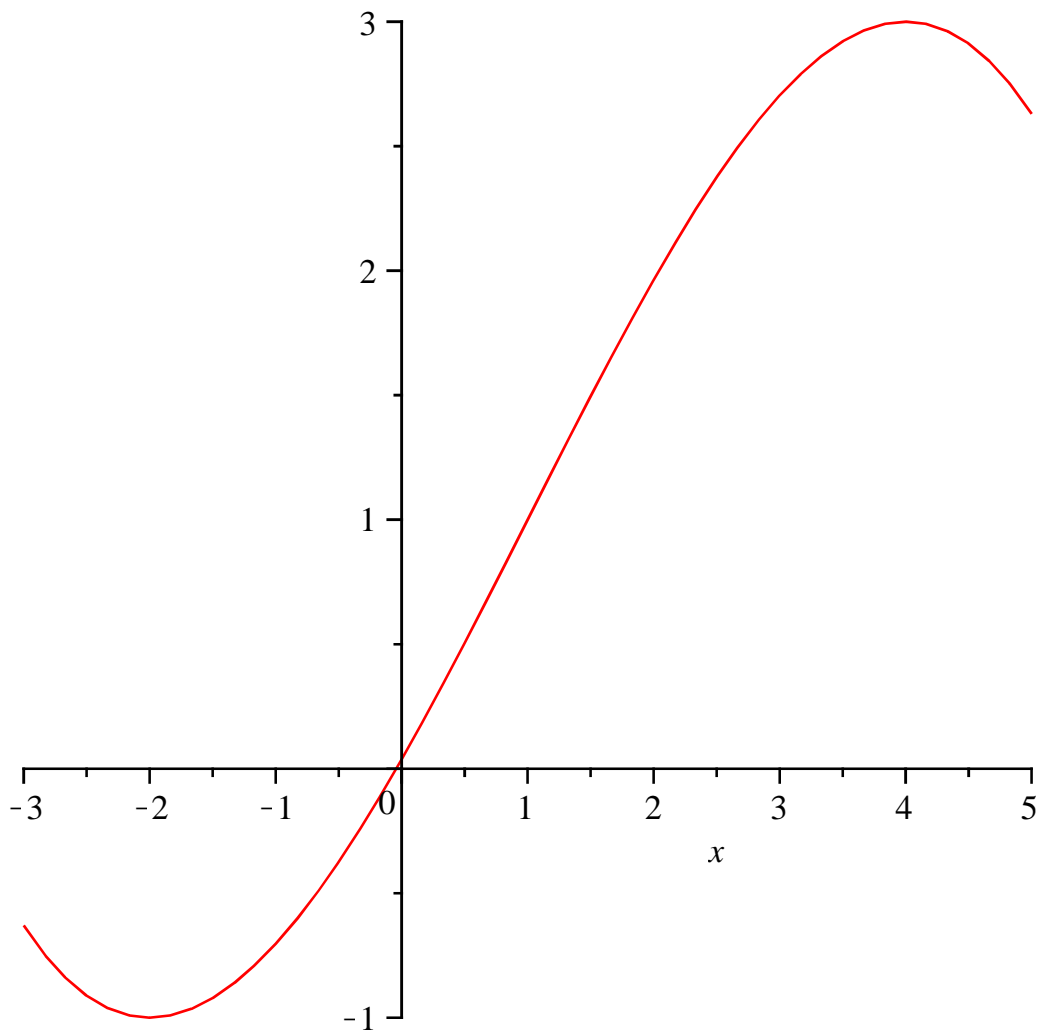
$$\begin{aligned} > eq4 := D(p)(4) = 0; \\ & \qquad \qquad \qquad eq4 := 48 a + 8 b + c = 0 \end{aligned} \tag{9}$$

$$\begin{aligned} > sol := \text{solve}(\{eq1, eq2, eq3, eq4\}, \{a, b, c, d\}); \\ & \qquad \qquad \qquad sol := \left\{ a = -\frac{1}{27}, b = \frac{1}{9}, c = \frac{8}{9}, d = \frac{1}{27} \right\} \end{aligned} \tag{10}$$

$$\begin{aligned} > psol := \text{subs}(sol, p(x)); \\ & \qquad \qquad \qquad psol := -\frac{1}{27} x^3 + \frac{1}{9} x^2 + \frac{8}{9} x + \frac{1}{27} \end{aligned} \tag{11}$$

So, the above polynomial should do it. Let's graph it.

$$> \text{plot}(psol, x = -3 .. 5);$$

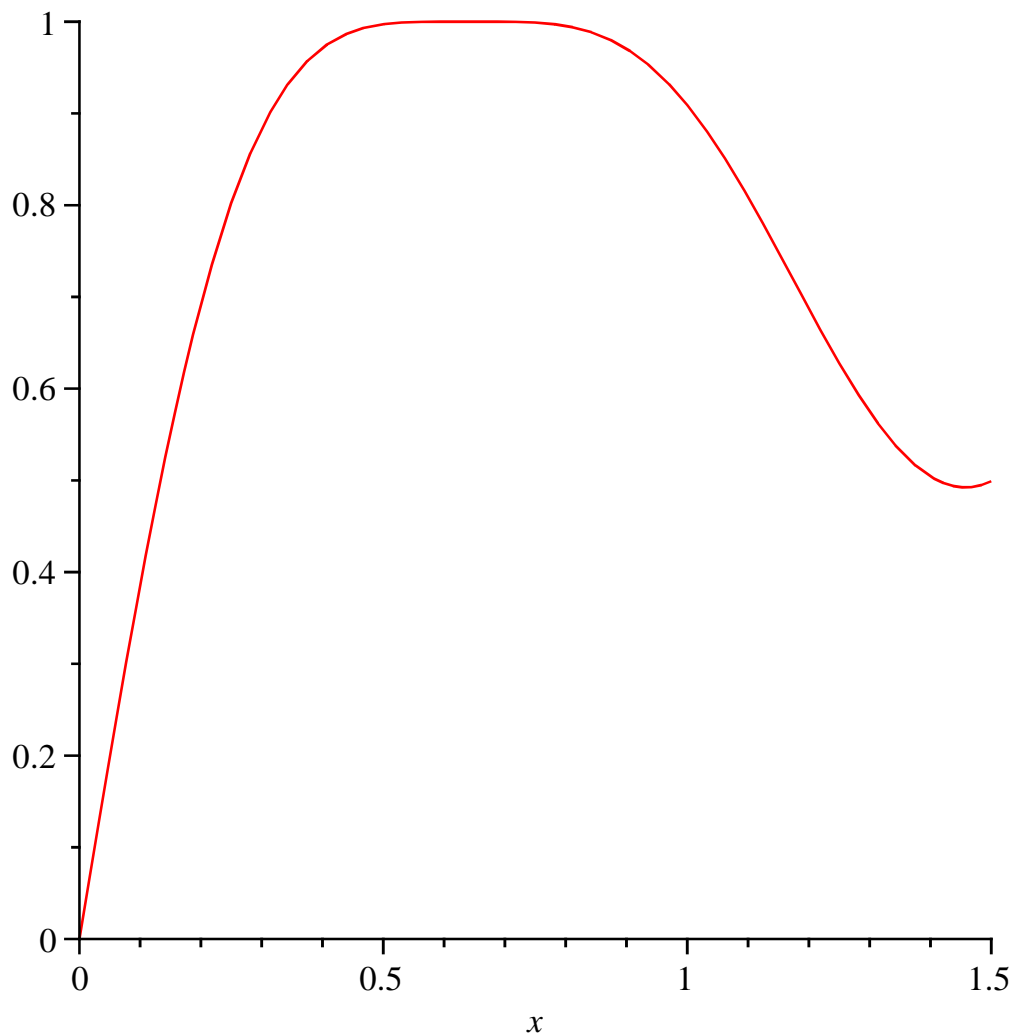


Looks good!

Problem 3:

```
> restart;  
> f := x → sin(x + sin(3·x));  
f := x → sin(x + sin(3 x))  
> plot(f(x), x = 0 .. 1.5);
```

(12)



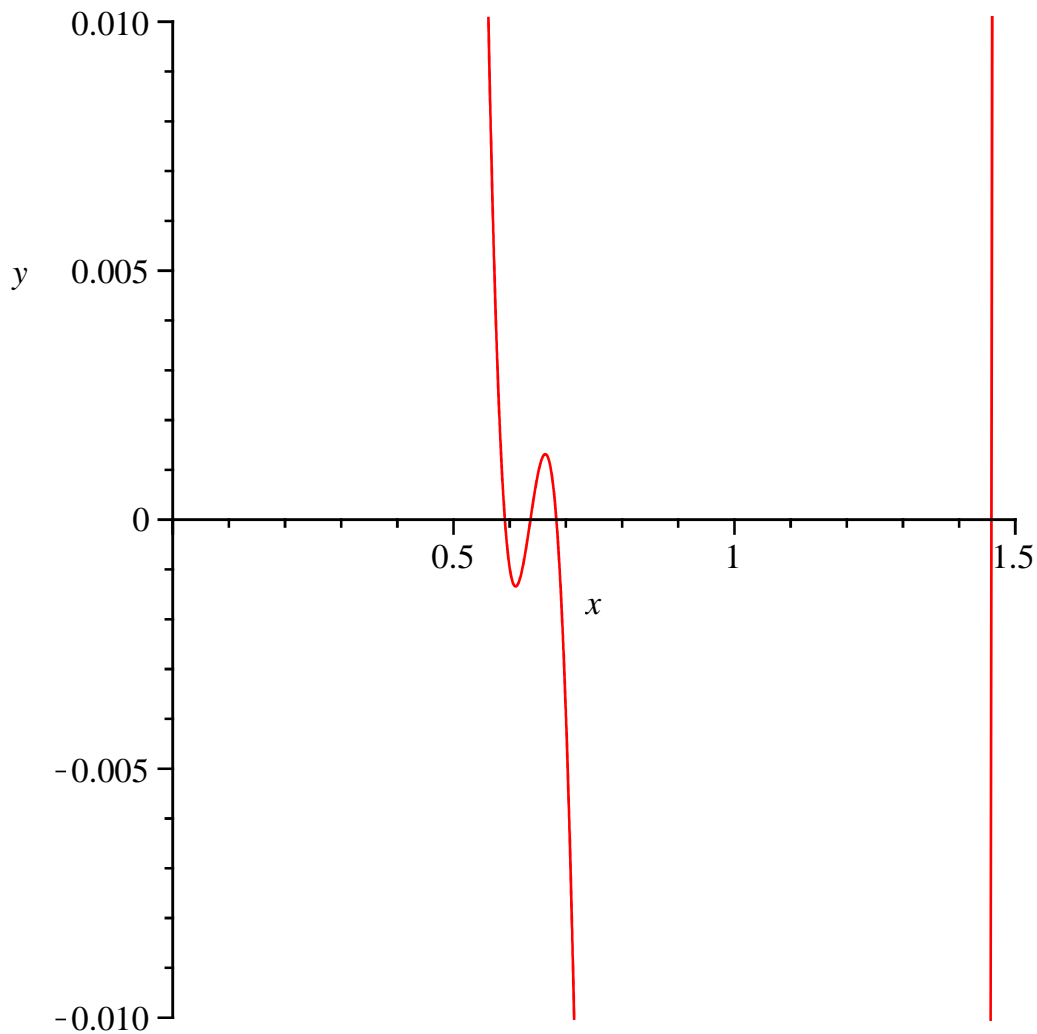
It looks like there's one interior maximum and one interior minimum. However, that maximum looks suspiciously broad. Let's look at the derivative, restricting our attention to values very close to zero.

```

> Df := D(f);
                                Df := x → cos(x + sin(3 x)) (1 + 3 cos(3 x))
> plot(Df(x), x = 0 .. 1.5, y = -0.01 .. 0.01, numpoints = 5000);

```

(13)



In fact, there are two interior maxima and two interior minima.

$$\begin{aligned} > \text{max1} := \text{fsolve}(Df(x) = 0, x = 0.5..0.6); \\ & \text{max1} := 0.5914295288 \end{aligned} \tag{14}$$

$$\begin{aligned} > \text{min1} := \text{fsolve}(Df(x) = 0, x = 0.6..0.65); \\ & \text{min1} := 0.6368777454 \end{aligned} \tag{15}$$

$$\begin{aligned} > \text{max2} := \text{fsolve}(Df(x) = 0, x = 0.65..1); \\ & \text{max2} := 0.6830700323 \end{aligned} \tag{16}$$

$$\begin{aligned} > \text{min2} := \text{fsolve}(Df(x) = 0, x = 1..1.5); \\ & \text{min2} := 1.457517357 \end{aligned} \tag{17}$$

The interior maxima are:

$$\begin{aligned} > [\text{max1}, f(\text{max1})]; \\ & [0.5914295288, 1.] \end{aligned} \tag{18}$$

$$\begin{aligned} > [\text{max2}, f(\text{max2})]; \\ & [0.6830700323, 1.] \end{aligned} \tag{19}$$

The interior minima are:

$$\begin{aligned} > [\text{min1}, f(\text{min1})]; \\ & [0.6368777454, 0.9999604801] \end{aligned} \tag{20}$$

$$\begin{aligned} > [\text{min2}, f(\text{min2})]; \\ & \end{aligned} \tag{21}$$

Remarkably for such an innocent looking function, it decreases from a local max of 1 all the way down to a local min of 0.9999604801, and then increases back to a local max of 1. Whether you include endpoints as local maxes or mins depends on which calculus book you use, so I didn't require it.

Problem 4: Even though we have some powerful computing abilities on our side, we still have to think before we charge ahead. Suppose that a line is tangent to a curve $y=f(x)$ at two points $(a, f(a))$ and $(b, f(b))$. What is the slope of the line? It can be written as the derivative at a , the derivative at b , or as $\frac{f(b) - f(a)}{b - a}$. These must all be equal, which will give us what we need.

$$\begin{aligned} & \text{restart;} \\ & f := x \rightarrow x^4 + x^3 - x^2 - x; \end{aligned} \quad f := x \rightarrow x^4 + x^3 - x^2 - x \quad (22)$$

$$\begin{aligned} & eq1 := D(f)(a) = D(f)(b); \\ & eq1 := 4a^3 + 3a^2 - 2a - 1 = 4b^3 + 3b^2 - 2b - 1 \end{aligned} \quad (23)$$

$$\begin{aligned} & eq2 := D(f)(a) = \frac{f(b) - f(a)}{b - a}; \\ & eq2 := 4a^3 + 3a^2 - 2a - 1 = \frac{b^4 + b^3 - b^2 - b - a^4 - a^3 + a^2 + a}{b - a} \end{aligned} \quad (24)$$

$$\begin{aligned} & sol := solve(\{eq1, eq2, a < b\}, \{a, b\}); \\ & sol := \left\{ a = -\frac{1}{4} - \frac{1}{4}\sqrt{11}, b = -\frac{1}{4} + \frac{1}{4}\sqrt{11} \right\} \end{aligned} \quad (25)$$

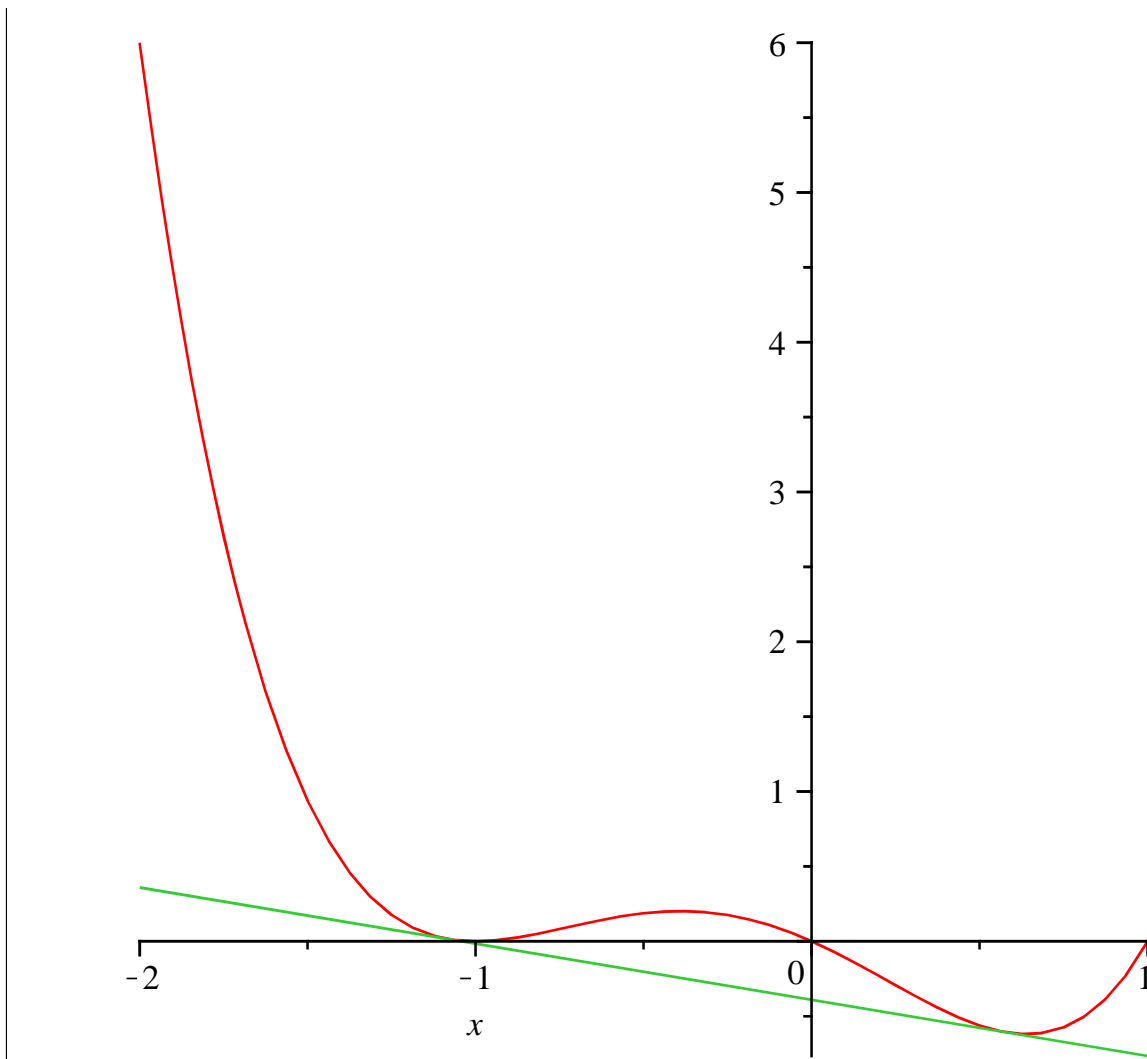
$$\begin{aligned} & asol := subs(sol, a); \\ & asol := -\frac{1}{4} - \frac{1}{4}\sqrt{11} \end{aligned} \quad (26)$$

$$\begin{aligned} & bsol := subs(sol, b); \\ & bsol := -\frac{1}{4} + \frac{1}{4}\sqrt{11} \end{aligned} \quad (27)$$

Now that we have the x coordinates of the points, it's easy to draw the line. I'll put it on the same plot as the original function to check that it makes sense. First, here's the line in an easier way to understand.

$$\begin{aligned} & y = f(asol) + (x - asol) \cdot D(f)(asol); \text{simplify}(\%); \\ & y = \left(-\frac{1}{4} - \frac{1}{4}\sqrt{11} \right)^4 + \left(-\frac{1}{4} - \frac{1}{4}\sqrt{11} \right)^3 - \left(-\frac{1}{4} - \frac{1}{4}\sqrt{11} \right)^2 + \frac{1}{4} + \frac{1}{4}\sqrt{11} + \left(x \right. \\ & \quad \left. + \frac{1}{4} + \frac{1}{4}\sqrt{11} \right) \left(4 \left(-\frac{1}{4} - \frac{1}{4}\sqrt{11} \right)^3 + 3 \left(-\frac{1}{4} - \frac{1}{4}\sqrt{11} \right)^2 - \frac{1}{2} + \frac{1}{2}\sqrt{11} \right) \\ & \quad y = -\frac{25}{64} - \frac{3}{8}x \end{aligned} \quad (28)$$

$$\text{plot}(\{f(x), f(asol) + (x - asol) \cdot D(f)(asol)\}, x = -2 .. 1);$$



Problem 4.

Here's the ODE:

```
> restart;
```

$$\text{ode} := \text{diff} \left(\frac{r \cdot \text{diff}(u(r), r)}{\text{sqrt}(1 + (\text{diff}(u(r), r)^2))}, r \right) = r \cdot u(r);$$

$$\text{ode} := \frac{\frac{d}{dr} u(r)}{\sqrt{1 + \left(\frac{d}{dr} u(r)\right)^2}} + \frac{r \left(\frac{d^2}{dr^2} u(r)\right)}{\sqrt{1 + \left(\frac{d}{dr} u(r)\right)^2}} - \frac{r \left(\frac{d}{dr} u(r)\right)^2 \left(\frac{d^2}{dr^2} u(r)\right)}{\left(1 + \left(\frac{d}{dr} u(r)\right)^2\right)^{3/2}} = r u(r) \quad (29)$$

```
> a := 0.53695;
```

$$a := 0.53695 \quad (30)$$

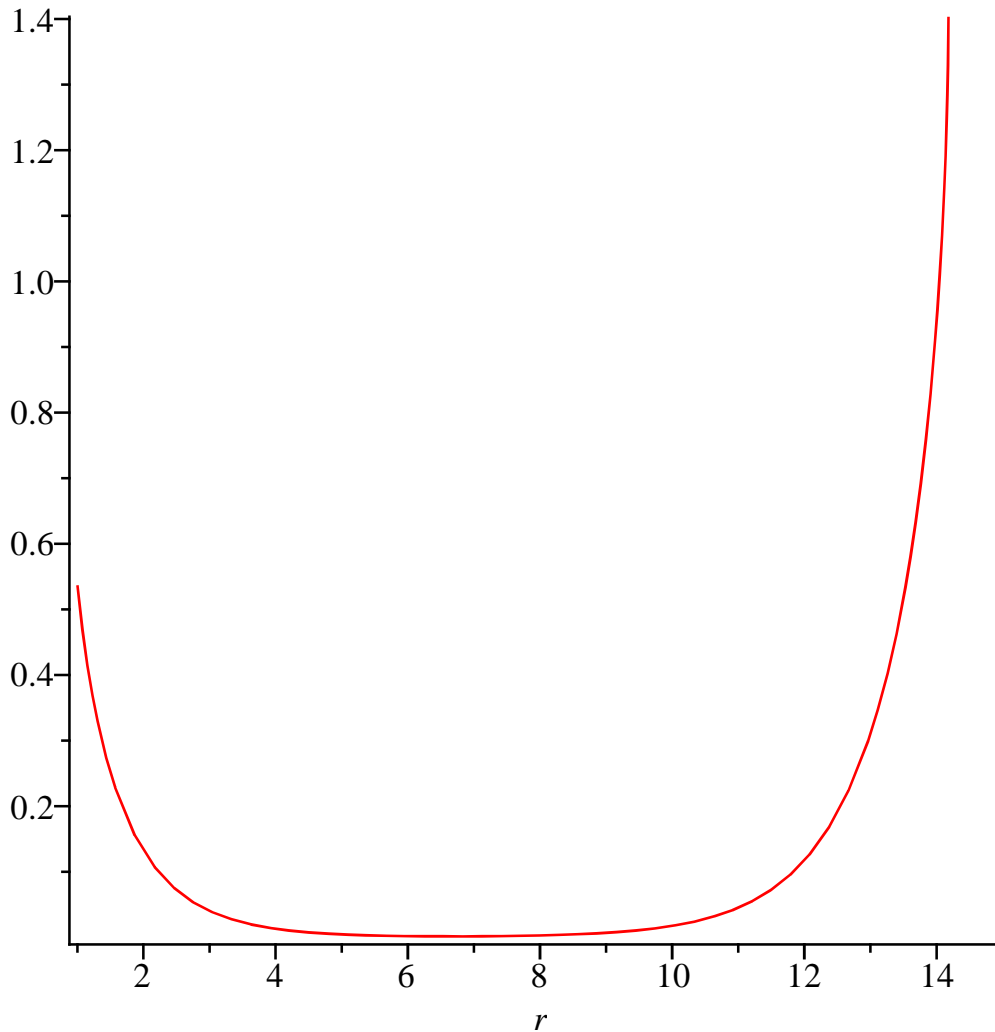
```
> dsol := dsolve( {ode, u(1) = a, D(u)(1) = -1}, u(r), numeric, output = listprocedure);
```

$$\text{dsol} := \left[r = \text{proc}(r) \dots \text{end proc}, u(r) = \text{proc}(r) \dots \text{end proc}, \frac{d}{dr} u(r) = \text{proc}(r) \dots \text{end proc} \right] \quad (31)$$

```
> usol := subs(dsol, u(r));
                               usol := proc(r) ... end proc (32)
```

The approach is to change the value of a. Too large and the solution eventually blows up; too small and the solution goes to minus infinity. Below, I have the solution for a=0.53695, which is too large. But if you plot the solution for a=0.53694, it goes to minus infinity. So, to four decimals, the answer is 0.5369. (You have to go to the next decimal to make sure that the answer doesn't round up to 0.5370.) (Actually, it's not obvious that the curves behave as we're assuming: that for different values of a, the curves can't cross. If you have too much time on your hands, you might think about proving it.)

```
> plot(usol(r), r = 1 .. 15);
```



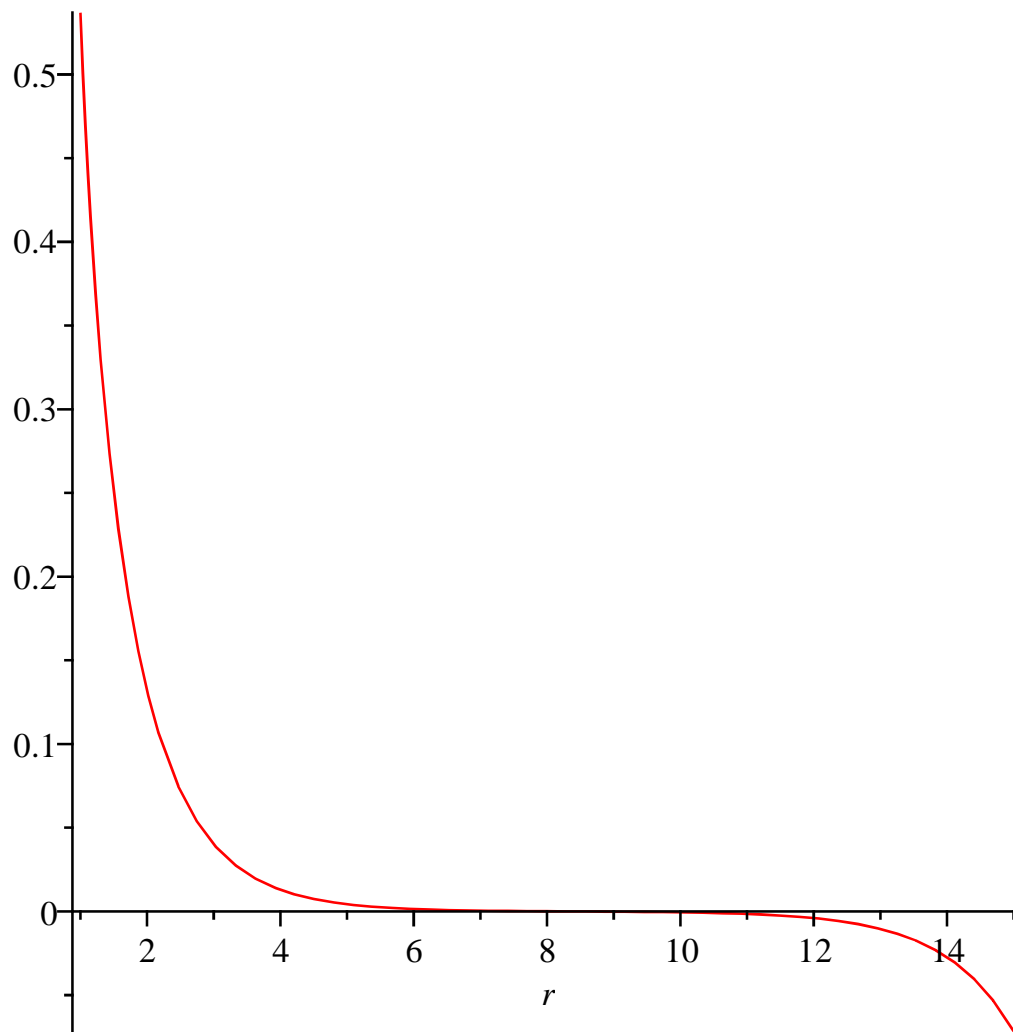
```
> a := 0.53694;
                               a := 0.53694 (33)
```

```
> dsol := dsolve( {ode, u(1) = a, D(u)(1) = -1 }, u(r), numeric, output = listprocedure);
dsol := [r = proc(r) ... end proc, u(r) = proc(r) ... end proc, d/d r u(r) = proc(r) (34)
...
end proc]
```

```
> usol := subs(dsol, u(r));
                               usol := proc(r) ... end proc (35)
```

Here's the plot for a = 0.53694.

```
> plot(usol(r), r=1..15);
```



```
[ ]  
[ ]  
[ ]
```