1 Dimensions

In general, a *dimension* is any quantity which can be measured. If we want to compute the average velocity of an object, we typically measure two quantities: the distance the object traveled and the amount of time the object was moving. The dimensions of velocity are then length L, divided by time T, and write

$$\text{dimensions of velocity} = [v] = LT^{-1},$$

where length and time are regarded as *fundamental* dimensions and velocity is regarded as a derived dimension. According to the International System of Units (SI), there are seven fundamental dimensions, which I list in the following table, along with their associated base units.

Dimension	Base SI unit
length L	meter (m)
mass M	kilogram (kg)
time T	second (s)
temperature Θ	kelvin (K)
electric current E	ampere (A)
luminous intensity I	candela (cd)
amount of substance A	mole (mol)

In dealing with population dynamics, we will often refer to population as *biomass* and treat this as an eighth fundamental dimension B. Typical physical quantities and their associated dimensions are listed in the next table

Quantity	Dimensions
Length	L
Time	T
Mass	M
Velocity	LT^{-1}
Acceleration	LT^{-2}
Force	MLT^{-2}
Energy	ML^2T^{-2}
Momentum	MLT^{-1}
Work	ML^2T^{-2}

Quantity	Dimensions
Frequency	T^{-1}
Density	ML^{-3}
Angular momentum	ML^2T^{-1}
Viscosity	$ML^{-1}T^{-1}$
Pressure	$ML^{-1}T^{-2}$
Power	ML^2T^{-3}
Entropy	ML^2T^{-2}
Heat	ML^2T^{-2}
Momentum	MLT^{-1}

2 Dimensional analysis and the Buckingham Pi theorem

Physical laws shouldn't change if you change the units in which the quantities are measured. For example, the area of a rectangle will be the base times the height regardless of whether we measure the distances in feet or meters, as long as the units of area are square feet or square meters. As another example, the period of oscillation t of a simple pendulum of length l, assuming a small initial oscillation is

$$t = 2\pi\sqrt{\frac{l}{g}} \tag{1}$$

which is valid if length is measured in feet, meters, or miles, and time is measured in seconds, minutes, or hours, as long as you're consistent. (The gravitational constant g has units as well: $g = 32\text{ft/sec}^2 = 9.8\text{m/sec}^2$, etc.) However, if you substitute in $g = 32$ and get $t = 1.11\sqrt{l}$, you get an equation which is only valid when length is measured in feet, and time is measured in seconds. The first equation is *dimensionally homogeneous,* and the second is not. An equation is dimensionally homogeneous if all terms in it have the same units. All physical laws may be made dimensionally homogeneous by assigning the correct units to physical constants. This is pretty much common sense: we all know that adding one term with units of length to another with units of mass is unlikely to be helpful. Dimensionally, equation (1) makes sense: the dimension of t is T, the dimension of l is L, the dimensions of g are LT^{-2}, and π is dimensionless.

A general note: it doesn't make sense to plug a quantity with physical dimension into a transcendental function (like sine, e^x, etc.). The reason is clear if you consider the Taylor expansion. For example, what would e raised to the 1 ft power mean?

$$e^{1ft} = 1 + 1ft + \frac{1}{2}\left(1ft\right)^2 + \frac{1}{3!}\left(1ft\right)^3 + \cdots,$$

where the sum on the right makes no sense dimensionally.

The basic theorem of dimensional analysis is the **Buckingham Pi theorem:** an equation is dimensionally homogeneous if and only if it can be written as $f(\Pi_1, \cdots, \Pi_k) = 0$, where f is some function and the Π_i's are dimensionless products and quotients of the variables and physical constants appearing in the problem. We don't need all such products, only a set from which all others may be formed. For example, equation (1) may be written as

$$\frac{tg^{1/2}}{l^{1/2}} - 2\pi = 0,$$

where all terms are dimensionless. (The number π isn't related to the Π's in the Pi Theorem.)

Example: Let's see how close dimensional analysis will get us to the period of a pendulum. Suppose that we decide that the period t might depend on length l, the acceleration g due to gravity, and m, the mass of the pendulum, but that no other quantities can effect the period (in particular, suppose that we've observed that the initial angle doesn't seem to effect the period). So, t has dimension T, l has dimension L, m has dimension M, and g has dimension LT^{-2}. The relation connecting these must be dimensionally homogeneous. How can we get a dimensionless product from these to use in the Pi theorem? A product $\Pi = t^a l^b g^c m^d$ has units of $T^{a-2c}L^{b+c}M^d$, so we must have $a = 2c$, $b = -c$, and $d = 0$.

Thus $\Pi = (t^2 l^{-1} g)^c$, so essentially the only dimensionless way of combining them is as $t^2 l^{-1} g$. Therefore, a physical law relating t, l, g, and m must be of the form $f\left(\dfrac{t^2 g}{l}\right) = 0$. Notice that it is dimensionally impossible that the period of a pendulum depend on the mass of the pendulum! Even including the starting angle wouldn't change that, since it wouldn't have dimensions involving mass (although if we include air resistance, we could involve mass of the pendulum). Solving $f\left(\dfrac{t^2 g}{l}\right) = 0$ for Π (which we should be able to do unless f is degenerate) gives us that $\dfrac{t^2 g}{l} = k$ for some dimensionless constant k, i.e., that $t = k\sqrt{\dfrac{l}{g}}$ for some dimensionless k. Dimensional analysis gives us no idea what k might be, however. If we were approaching this experimentally, we might observe a bunch of periods for different lengths, compute $t\sqrt{\dfrac{g}{l}}$, and see what number these are close to. In problem 1, you will look at a more general pendulum model.

Example: Suppose we fire a projectile straight up. Let's apply dimensional analysis to the question of how far up it will go. Assume that the height remains small enough so that the acceleration due to gravity is essentially constant (we're aren't firing it into orbit or anything), and neglect air resistance.

Solution: Call the maximum height h. What might this depend on? Certainly the initial velocity v_0 comes into play. Also the acceleration due to gravity g is involved. I can't think of anything else. A product $\Pi = h^a (v_0)^b g^c$ has units $L^a \left(LT^{-1}\right)^b \left(LT^{-2}\right)^c = L^{a+b+c} T^{-b-2c}$. For this to be dimensionless, we need

$$\begin{aligned} a + b + c &= 0 \\ -b - 2c &= 0. \end{aligned}$$

There's one free variable; let's take it to be c. Then $b = -2c$ and $a = c$. Thus $\Pi = \left(h\,(v_0)^{-2}\, g\right)^c$, so essentially the only dimensionless way of combining these is as $\frac{hg}{v_0^2}$. Thus a physical law involving these must be of the form $f\left(\frac{hg}{v_0^2}\right) = 0$. Solve this for Π to get $\frac{hg}{v_0^2} = k$ for some dimensionless k, so that we must have

$$h = k\frac{v_0^2}{g}.$$

This is as far as dimensional analysis can take us. Using calc I, you can find that $k = \frac{1}{2}$.