More Dimensional Analysis examples

Famous example: In 1947, high speed photographs of an early A-bomb test at Alamagordo were declassified, and published in *Life* magazine. They gave the radius of the shockwave at different times. Using these, British and Soviet scientists were able to estimate the yield of the explosion, which was then still classified. Here's the analysis. Assume that the relevant variables are the bomb yield, i.e., the total amount E of energy released, the radius r of the shockwave, the time t, and the air density ρ of the surrounding undisturbed air. (Obviously, the hardest part of this sort of analysis is deciding what variables are relevant. In problems and such, you'll generally be told which variables to consider.) The units are: $[E] = ML^2T^{-2}$, $[r] = L$, $[t] = T$, $[\rho] = ML^3$. A product $\Pi = E^a r^b t^c \rho^d$ has units

$$\left(ML^2T^{-2}\right)^a L^b T^c \left(ML^3\right)^d = M^{a+d} L^{2a+b+3d} T^{-2a+c}.$$

This will be dimensionless iff

$$\begin{aligned} a + d &= 0 \\ 2a + b - 3d &= 0 \\ -2a + c &= 0. \end{aligned}$$

So, we row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & | & 0 \\ 2 & 1 & 0 & -3 & | & 0 \\ -2 & 0 & 1 & 0 & | & 0 \end{array}\right]$$

and get

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & -5 & | & 0 \\ 0 & 0 & 1 & 2 & | & 0 \end{array}\right],$$

which says $c = -2d$, $b = 5d$, $a = -d$, which says that Π must be $\left(E^{-1}r^5t^{-2}\rho\right)^d$. We can take d to be 1, so a physical law involving these quantities must be of the form

$$f(\Pi) = f\left(E^{-1}r^5t^{-2}\rho\right) = 0.$$

Solving for Π,

$$E^{-1}r^5t^{-2}\rho = k,$$

where k is a dimensionless constant. By making experiments with dynamite (don't try this at home!), a good estimate for k can be found. Since radius and time were given in the photos, and the density of air is known, the yield of the explosion may be estimated, and it turned out to be pretty accurate.

Example: One often sees in ODE's books the assumption that air resistance (or resistance to motion through fluid) is proportional to either the speed of the object or the square of the speed of the object. It seems likely that dimensionally one of these is wrong. Let's analyze this. Suppose that we have a fixed ball in a uniform flow which exerts a drag force F. The relevant quantities are the upstream velocity v, the radius r of the ball, the viscosity μ of the liquid, and the density ρ of the liquid. The units of F are MLT^{-2}, the units of v are LT^{-1}, the units of r are L, the units of μ turn out to be $ML^{-1}T^{-1}$, and the units of ρ are ML^{-3}. We seek dimensionless products of these quantities. If

$$\Pi = F^a V^b r^c \mu^d \rho^e,$$

we look at mass, length, and time to get this system:

$$\begin{aligned} a + d + e &= 0 \\ a + b + c - d - 3e &= 0 \\ -2a - b - d &= 0 \end{aligned}$$

Row reducing, we find that the general solution may be written as: d and e are arbitrary, $a = -d - e$, $b = d + 2e$, and $c = d + 2e$. Since we have two arbitrary variables, we may write every possible dimensionless product in terms of

$$\Pi_1 = \frac{vr\mu}{F}$$

and

$$\Pi_2 = \frac{v^2r^2\rho}{F},$$

where Π_1 is obtained by setting $d = 1$ and $e = 0$, and Π_2 is obtained by setting $d = 0$ and $e = 1$. To get this in terms of quantities that are more commonly used, look at $\frac{\Pi_2}{\Pi_1}$ and $\frac{1}{\Pi_2}$. Clearly every dimensionless product can be written in terms of these as well. The first is called the Reynolds number of the flow $\frac{vr\rho}{\mu}$, (where r is a representative length) and the second is the pressure coefficient of the flow $\frac{F}{v^2r^2\rho}$.

By the Pi theorem, any physical relation between the five given quantities can be written as

$$f\left(\frac{F}{v^2r^2\rho}, \frac{vr\rho}{\mu}\right) = 0,$$

or solving this for the first variable,

$$\frac{F}{v^2r^2\rho} = g\left(\frac{vr\rho}{\mu}\right),$$

i.e,

$$F = v^2r^2\rho g\left(\frac{vr\rho}{\mu}\right).$$

Depending on the form of g, we could get any power of v, so dimensional analysis can't completely resolve the initial question. However, if we can make a definite statement in the limit as the Reynolds number goes to zero. If the

Reynolds number is small (creeping flow), we get that F is proportional to v^2. Also, viscosity of air is very tiny, essentially zero, and one can start with the assumption that the viscosity doesn't enter into it. If we take μ out of the analysis, we're looking for dimensionless Π's of the form $F^a V^b r^c \rho^e$. Setting d to be zero in the system which we solved gives us only one dimensionless way of doing this: $\Pi = \frac{v^2 r^2 \rho}{F}$. Solving $f(\Pi) = 0$ for Π, we get $\frac{v^2 r^2 \rho}{F} = k$, so the drag force F is proportional to v^2 in this case as well. In general, the assumption that drag is proportional to the square of the speed of the flow is more realistic that assuming that it's proportional to the first power of speed. (It turns out that the first power is more accessible mathematically, which is why it comes up in elementary ODE's texts.