Dimensional analysis problems
Math 647.600

1. Why do stringed musical instruments have strengths of different lengths and thicknesses? Assume that the fundamental frequency ω of vibration of a string depends on its length l, mass per unit length μ, and tension (force) F on the string. Prove that ω must be proportional to $\frac{\sqrt{F/\mu}}{l}$.

   Answer: The physical quantities involved are frequency ω, mass per unit length μ, length l, and tension F. The dimensions of these are: $[\omega] = T^{-1}$, $[\mu] = ML^{-1}$, $[l] = L$, and $[F] = MLT^{-2}$, where the dimensions of frequency and force are given in the class notes. A product of these is $\Pi = \omega^a \mu^b l^c F^d$, which has dimensions

$$\begin{aligned}[\Pi] &= T^{-a}\left(ML^{-1}\right)^b L^c \left(MLT^{-2}\right)^d \\ &= T^{-a-2d} M^{b+d} L^{-b+c+d}.\end{aligned}$$

   This will be dimensionless if and only if

$$\begin{aligned}-a-2d &= 0 \\ b+d &= 0 \\ -b+c+d &= 0.\end{aligned}$$

   The augmented matrix for this system is

$$\left(\begin{array}{cccc:c} -1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \end{array}\right),$$

   which row reduces to

$$\left(\begin{array}{cccc:c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right).$$

   The interpretation of this is that d is arbitrary, $c = -2d$, $b = -d$, and $a = -2d$. Thus the only possibility for a dimensionless Π is

$$\Pi = \left(\omega^{-2}\mu^{-1}l^{-2}F\right)^d.$$

   The arbitrary power d may be ignored, since a function of $\left(\omega^{-2}\mu^{-1}l^{-2}F\right)^d$ is also a function of $\omega^{-2}\mu^{-1}l^{-2}F$. Thus a physical law relating these quantities must be of the form

$$f\left(\omega^{-2}\mu^{-1}l^{-2}F\right) = 0,$$

which we may generically solve for Π to get

$$\omega^{-2}\mu^{-1}l^{-2}F = k,$$

for some dimensionless constant k. Thus

$$\omega = \sqrt{\frac{kF}{\mu l^2}} = k'\frac{\sqrt{F/\mu}}{l},$$

as desired.

2. We now want to include frictional effects and the initial angle in analyzing pendulums. As in the lecture notes, the length of the pendulum is l and its mass is m.

   (a) Suppose that the frictional force is due primarily to air and is proportional to v^2 with constant of proportionality k. Let τ be the time required for the pendulum to reach half its initial amplitude θ. Determine the dimensions of k. Show that

   $$\tau = \sqrt{\frac{l}{g}}G\left(\theta, \frac{kl}{m}\right)$$

   for some function G.

   Answer: The quantities involved are τ, length l, mass m, initial amplitude θ, acceleration due to gravity g, and the drag constant k. The dimensions of all but the last quantity are clear. If drag force is k times v^2, then the dimensions of that equation are $MLT^{-2} = [k]\,L^2T^{-2}$, so k must have dimensions of ML^{-1}. A product Π of the six quantities involved is

   $$\Pi = \tau^a l^b m^c \theta^d g^e k^f,$$

   which has dimensions

   $$\begin{aligned}[\Pi] &= T^a L^b M^c \left(LT^{-2}\right)^e \left(ML^{-1}\right)^f \\ &= T^{a-2e}L^{b+e-f}M^{c+f}.\end{aligned}$$

   This will be dimensionless if and only if

   $$\begin{aligned} a - 2e &= 0 \\ b + e - f &= 0 \\ c + f &= 0. \end{aligned}$$

   (d does not appear, of course, since θ has no dimension.) The augmented matrix for the system is

   $$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right),$$

which is already in reduced row echelon form. The interpretation is that c, d, and e are arbitrary, that $c = -f$, $b = -e + f$, and that $a = 2e$. Thus a dimensionless product must be of the form

$$\begin{aligned} \Pi &= \tau^{2e} l^{-e+f} m^{-f} \theta^d g^e k^f \\ &= \theta^d \left(\tau^2 l^{-1} g\right)^e \left(l m^{-1} k\right)^f . \end{aligned}$$

Thus any physical law relating these quantities must be of the form

$$F\left(\theta, \frac{\tau^2 g}{l}, \frac{kl}{m}\right) = 0.$$

Generically, we may solve this equation for the middle variable in terms of the rest:

$$\frac{\tau^2 g}{l} = H\left(\theta, \frac{kl}{m}\right),$$

so

$$\begin{aligned} \tau &= \sqrt{\frac{l}{g}} \sqrt{H\left(\theta, \frac{kl}{m}\right)} \\ &= \sqrt{\frac{l}{g}} G\left(\theta, \frac{kl}{m}\right), \end{aligned}$$

where G is some unknown function of two variables. (The point of the dimensional analysis is that without it, all we could say is that τ is some unknown function of the other five variables.)

(b) Deduce a similar result if the frictional force is assumed to be proportional to v.

Answer: Of course, the dimensions of k will change. If drag force is kv, then the dimensions of that equation are $MLT^{-2} = [k]\, LT^{-1}$, and the dimensions of k are MT^{-1}. A product Π of the six quantities involved is

$$\Pi = \tau^a l^b m^c \theta^d g^e k^f,$$

which has now dimensions

$$\begin{aligned} [\Pi] &= T^a L^b M^c \left(LT^{-2}\right)^e \left(MT^{-1}\right)^f \\ &= T^{a-2e-f} L^{b+e} M^{c+f}. \end{aligned}$$

Now we need

$$\begin{aligned} a - 2e - f &= 0 \\ b + e &= 0 \\ c + f &= 0 \end{aligned}$$

to have Π dimensionless. This has augmented matrix

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & -2 & -1 & \vdots & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & \vdots & 0 \end{array}\right).$$

Again, this turns out to already be in reduced row echelon form. The variables d, e, and f are arbitrary, and $c = -f$, $b = -e$, and $a = 2e + f$. So now we can say that a dimensionless product must be

$$\begin{aligned} \Pi &= \tau^{2e+f} l^{-e} m^{-f} \theta^d g^e k^f \\ &= \theta^d \left(\tau^2 l^{-1} g\right)^e \left(\tau m^{-1} k\right)^f . \end{aligned}$$

This is giving us three dimensionless products of the quantities: $\Pi_1 = \theta$, $\Pi_2 = \frac{\tau^2 g}{l}$, and $\Pi_3 = \frac{\tau k}{m}$, and any physical law must be a function of these three. Unfortunately, τ appears in two of them, so we'll have to be a bit sneakier than in part a and come up with a different collection of three dimensionless constants. Any function of Π_1, Π_2, and Π_3 may also be written as a function of Π_1, Π_2, and $\frac{\Pi_2}{\Pi_3^2} = \frac{gm^2}{lk^2}$ and vice-versa. So, instead of the original three Π's we found, take the three dimensionless quantities θ, $\frac{\tau^2 g}{l}$, and $\frac{gm^2}{lk^2}$ to describe any physical law relating the variables in the problem. As in part a, we must have a physical law being of the form

$$F\left(\theta, \frac{\tau^2 g}{l}, \frac{gm^2}{lk^2}\right) = 0,$$

and solving for the middle variable as above,

$$\tau = \sqrt{\frac{l}{g}} G\left(\theta, \frac{gm^2}{lk^2}\right)$$

3. How long should you roast a turkey? Typically, cookbooks give instructions such as: "set the oven to T_0 degrees and allow n minutes per pound for cooking." This is saying that the cooking time is proportional to the mass of the turkey. Does this make sense from the point of view of dimensional analysis? A piece of meat is cooked when its minimum internal temperature reaches a certain value dependent on the type of meat and the desired doneness. The variables in the problem are: the cooking time t, the difference in temperature ΔT_r between the raw meat and the oven, the difference in temperature ΔT_c between the cooked meat and the oven, some characteristic dimension l of the meat, and the coefficient κ of heat conductance for the turkey. The dimensions of κ are L^2T^{-1}.

(a) Show that

$$t = \frac{l^2}{\kappa}\phi\left(\frac{\Delta T_r}{\Delta T_c}\right).$$

In particular, show that for geometrically similar turkeys differing only in size (i.e., the initial and desired temperatures are the same, along with density and coefficient of heat conductance), that the cooking time should be proportional to mass raised to the two-thirds power.

Answer: Take the variables which we are given: t, ΔT_r, ΔT_c, l, κ. These have dimensions T, Θ, Θ, L, and L^2T (recall that Θ stands for the dimension of temperature). A product $\Pi = t^a\left(\Delta T_r\right)^b\left(\Delta T_c\right)^c l^d \kappa^e$ has dimensions

$$\begin{aligned}[\Pi] &= T^a\Theta^b\Theta^c L^d\left(L^2T^{-1}\right)^e \\ &= T^{a-e}\Theta^{b+c}L^{d+2e},\end{aligned}$$

giving us the system

$$\begin{aligned}a - e &= 0 \\ b + c &= 0 \\ d + 2e &= 0\end{aligned}$$

with augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right).$$

Again, this happens to be in reduced row echelon form. We take c and e to be arbitrary, so that $d = -2e$, $b = -c$, and $a = e$. Thus for Π to be dimensionless,

$$\begin{aligned}\Pi &= t^e\left(\Delta T_r\right)^{-c}\left(\Delta T_c\right)^c l^{-2e}\kappa^e \\ &= \left(\frac{\Delta T_c}{\Delta T_r}\right)^c\left(\frac{\kappa t}{l^2}\right)^e.\end{aligned}$$

There are therefore two dimensionless Π's to consider: $\Pi_1 = \frac{\Delta T_r}{\Delta T_c}$ and $\Pi_2 = \frac{\kappa t}{l^2}$. (I flipped Π_1 to make it look like the answer we're supposed to get; it's no big deal.) So, by the Pi theorem, a physical law relating these must be of the form

$$F\left(\frac{\kappa t}{l^2}, \frac{\Delta T_c}{\Delta T_r}\right) = 0,$$

which we can generically solve for the first variable to get

$$\frac{\kappa t}{l^2} = \phi\left(\frac{\Delta T_c}{\Delta T_r}\right),$$

giving the desired relation. Now, what has this to do with cooking times of turkeys of different weights? Since we're assuming that turkeys are geometrically similar, the volume of a turkey will be proportional to l^3. We're also assuming that turkey density is constant, hence the weight of a turkey is proportional to its volume, hence mass is proportional to l^3. If initial and cooked temperatures are the same, as well as heat conductance, it follows that cooking time is proportional to l^2, hence to mass raised to the 2/3 power.

(b) Suppose we have a mutant 100 pound turkey, and a very, very big oven. How long should I cook it at 325°? Use the table at the National Turkey Federation's website:
http://www.eatturkey.com/consumer/cookinfo/turroast.html to estimate the constant of proportionality, explaining your reasoning. Assume that the turkey is not stuffed, and that we start it at refrigerator temperature, as they suggest.

Answer: We've decided in part a that cooking time is proportional to weight raised to the 2/3 power, assuming that all other variables are the same. There are a number of ways to use the cooking data given. Probably the most straight-forward is to take it as 6 data points, using the midpoint of the cooking times and the midpoint of the weight ranges. If cooking time satisfies $t = c \cdot w^{2/3}$, use these 6 points to estimate c. The simplest method is just to compute $t/w^{2/3}$ for all 5 and take their average. (I'm keeping 4 decimals, which is probably more than the accuracy of the given data really supports.) So, we have the following table:

weight (in lbs)	time (in hours)	$t/w^{2/3}$
10	2.875	0.6194
13	3.375	0.6104
16	4	0.6300
19	4.375	0.6144
22	4.75	0.6050
27	5.125	0.5694

The average of the third columns is 0.6081. The formula which we've come up with is that cooking time (in hours) should be 0.6081 times weight raised to the 2/3 power (in pounds). For a hundred pound turkey, that gives us 13.10 hours. Considering the ranges in the table, it's probably reasonable to say 12 3/4 hours to 13 1/4 hour. Giving more accuracy is not quite honest, since the data we're given is obviously not all that exact to begin with. If you do a least squares fit instead of averaging, you get 0.60178 for the constant. This gives 12.96 hours, leading to pretty much the same range of times.