CONVEX, ROTATIONALLY SYMMETRIC LIQUID BRIDGES BETWEEN SPHERES

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A liquid bridge between two balls will have a free surface which has constant mean curvature, and the angles of contact between the free surface and the fixed surfaces of the balls will be constant (although there might be two different contact angles: one for each ball). If we consider rotationally symmetric bridges, then the free surface must be a Delaunay surface, which may be classified as unduloids, nodoids, and catenoids, with spheres and cylinders as special cases of the first three types. In this paper, it is shown that a convex unduloidal bridge between two balls is a constrained local energy minimum for the capillary problem, and a convex nodoidal bridge between two balls is unstable.

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1. Introduction

The stability and energy minimality of a liquid bridge between parallel planes has been well studied (e.g., [**4**], [**9**], [**10**], [**13**], [**14**]). That of the related problem of a liquid bridge between fixed balls, illustrated in Figure 1, is less studied. (See however, [**1**], [**8**], and [**11**].) In this paper, we will give a simple way of determining if convex, rotationally symmetric bridges between fixed balls are energy minima. In fact, if a convex bridge between spheres is a section of an unduloid, it is a constrained local energy minimum, and if it is a section of a nodoid, it is unstable, and in particular not an energy minimum. For rotationally symmetric bridges, we will use "convex" to mean that the profile curve of the free surface is a convex function.

Someone familiar with [**10**] might be suspicious of the above claim. In [**10**] it is shown that convex bridges between planes are always stable. How could reducing the radius of the spheres from infinity to a finite amount change the behavior so drastically? The resolution of this apparent paradox is that in looking at bridges between parallel planes, one deals with stability

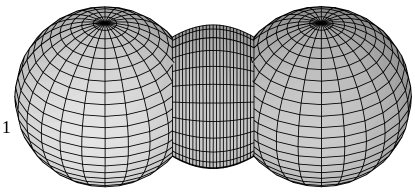

FIGURE 1. A liquid bridge between spheres

or energy minimality modulo translations parallel to the planes: there are perturbations which are automatically energy neutral. Changing the fixed surfaces from planes to spheres will change the boundary contribution of the relevant quadratic form ($\mathcal{M}$, as defined in (1.2)), in particular the value of the quadratic form as applied to the perturbations which were energy neutral for the bridge between planes. This is in fact the key point of the paper. If the bridge is a section of a nodoid, then in changing the fixed surfaces from planes to spheres, the energy neutral perturbations change to energy reducing perturbations, causing instability. On the other hand, if the bridge is a section of an unduloid, then in changing the fixed surfaces from planes to spheres, the energy neutral perturbations change to energy increasing ones, which we will show implies that the bridge is a constrained local energy minimum.

In considering the stability and energy minimality of a liquid bridge between solid balls, some concepts from the general theory of capillary surfaces must be recalled (see [**2**], [**12**], [**13**]). Suppose that Γ is the boundary of a fixed solid region in space, and that we put a drop of liquid in contact with Γ. Let Ω be the region in space occupied by the liquid, and Σ the free boundary of Ω (i.e., the part of $\partial\Omega$ not contained in Γ). In the absence of gravity or other external potentials, the shape of the drop results from minimizing the functional

$$\mathcal{E}(\Omega) = |\Sigma| - c|\Sigma_1|, \tag{1.1}$$

where $|\Sigma|$ is the area of the free surface of the drop, $|\Sigma_1|$ is the area of the region on Γ wetted by the drop, and $c \in [-1, 1]$ is a material constant. The minimization is under the constraint that the volume of the drop is fixed. The first order necessary conditions for a drop to minimize (1.1) are that the mean curvature of Σ is a constant H (this is a Lagrange multiplier arising from the volume constraint) and that the angle between the normals to Σ and to Γ along the curve of contact is constantly $\gamma = \arccos(c)$ (see [**2**]).

We will be referring to the following related concepts in this paper.

- A capillary surface Σ is a *constrained local energy minimum* if $\mathcal{E}(\Omega) < \mathcal{E}(\Omega')$ for any comparison drop Ω' which is near (but not the same as) Ω in an appropriate sense, and which contains the same volume of liquid. The question of what sense of "nearness" is appropriate is a complex one, but one approach which has been used is based on curvilinear coordinates ([**12**]).

- In the common special case that there is a group of symmetries which take Γ to itself, we say that Σ is a *constrained local energy minimum modulo symmetries* if $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega')$ for comparison drops Ω' which are near Ω, and if $\mathcal{E}(\Omega) = \mathcal{E}(\Omega')$, then Ω' is obtained by applying an element of the symmetry group to Ω. The specific example that we will deal with in this paper is that of a liquid bridge between parallel planes. No bridge could be a constrained local energy minimum, since translations parallel to the planes leave energy unchanged. However, in certain circumstances one can show that a given bridge is a minimum modulo these translations: any nearby bridge with the same energy (and volume) will be a translation of the original one ([**13**]).
- Suppose that $\Sigma = \Sigma(0)$ is embedded in a smoothly parameterized family of drops $\Sigma(\varepsilon)$, all of which contain the same volume. If $\frac{d^2}{d\varepsilon^2}\left(\mathcal{E}\left(\Sigma(\varepsilon)\right)\right)$ is negative at $\varepsilon = 0$ for that family, Σ is said to be *unstable*.
- If Σ is not unstable, then Σ is *stable*.

The quadratic form related to stability and energy minimality is

$$\mathcal{M}(\varphi, \varphi) = \iint_{\Sigma} |\nabla\varphi|^2 - |S|^2\varphi^2 \, d\Sigma + \oint_{\sigma} \rho\varphi^2 \, d\sigma. \tag{1.2}$$

Here $|S|^2$ is the square of the norm of the second fundamental form of Σ. (In terms of mean curvature H and Gaussian curvature K, $|S|^2$ may be written as $2(2H^2 - K)$, and in terms of the principal curvatures, $|S|^2$ may be written as $k_1^2 + k_2^2$.) We write σ for $\partial\Sigma$. The coefficient ρ is given by

$$\rho = \kappa_\Sigma \cot\gamma - \kappa_\Gamma \csc\gamma, \tag{1.3}$$

where κ_Σ is the curvature of the curve $\Sigma \cap \Pi$ and κ_Γ is the curvature of $\Gamma \cap \Pi$, if Π is a plane normal to the contact curve $\partial\Sigma$. These planar curvatures are signed: in Figure 2, both κ_Σ and κ_{Γ_n} are negative.

We will denote the subspace of $H^1(\Sigma)$ of all φ for which $\iint_\Sigma \varphi d\Sigma = 0$ by $1^\perp$, since this subspace is the collection of functions which are perpendicular to the constant function 1 in the H^1 inner product. The relationship between $\mathcal{M}$ and stability is that Σ is stable if and only if $\mathcal{M}(\varphi, \varphi) \geq 0$ for all $\varphi \in 1^\perp$. If Σ is a local energy minimum or a local energy minimum mod symmetries, then Σ is stable. However, stability does not imply that Σ is any sort of local energy minimum. It is not known whether the stronger condition $\mathcal{M}(\varphi, \varphi) > 0$ for all nontrivial $\varphi \in 1^\perp$ is enough to imply that a capillary surface is some sort of energy minimum. (See [**3**], [**12**] for discussion of this point. If the contact curves are "pinned" rather than free to move on Γ, then the strengthened condition will imply energy minimality ([**5**]).) In [**12**], it was shown that if for some $\varepsilon > 0$, we have $\mathcal{M}(\varphi, \varphi) \geq \varepsilon \|\varphi\|^2$ holding on $1^\perp$, where $\|\cdot\|$ is the $H^1(\Sigma)$ norm, then Σ is a volume constrained local minimum for energy. If $\mathcal{M}(\varphi,\varphi) \geq \varepsilon \|\varphi\|^2$ on a subspace for an $\varepsilon > 0$, $\mathcal{M}$ is said to be *strongly positive* on that subspace.

The quadratic form $\mathcal{M}$ is analyzed in [**12**] and [**13**] by considering an eigenvalue problem arising from integration by parts. Define the differential operator $\mathcal{L}$ by

$$\mathcal{L}(\psi) = -\Delta\psi - |S|^2\psi \tag{1.4}$$

where Δ is the Laplace-Beltrami operator on Σ. The eigenvalue problem we study is given by

$$\mathcal{L}(\psi) = \lambda\psi \tag{1.5}$$

on Σ, with

$$\mathbf{b}(\psi) \equiv \psi_1 + \rho\psi = 0 \tag{1.6}$$

on $\partial\Sigma$, where ψ_1 is the outward normal derivative of ψ. If the eigenvalue problem has no non-positive eigenvalues, the bridge is stable, and in fact a constrained local energy minimum. If there are two or more negative eigenvalues, then the bridge is unstable. If there is one negative eigenvalue, and the rest are positive, then there is a further condition which must be checked to see if the bridge is stable (see [**8**] and [**9**]). In [**13**] it is shown that a bridge between parallel planes must always have zero as a double eigenvalue, corresponding to energy neutral translations. The relationship between the bilinear form $\mathcal{M}$ and the operator $\mathcal{L}$ is that

$$\mathcal{M}(\varphi,\psi) = \iint_\Sigma \varphi\mathcal{L}(\psi)\, d\Sigma + \oint_\sigma \varphi\mathbf{b}(\psi)\, d\sigma, \tag{1.7}$$

after an integration by parts.

The above general theory must be modified when we consider bridges between fixed balls, at least when we want to allow for different contact angles on the different balls B_1 and B_2. In that case, there will be two material constants c_1 and c_2, and the energy functional will be

$$\mathcal{E}(\Omega) = |\Sigma| - c_1|\Sigma_1| - c_2|\Sigma_2|, \tag{1.8}$$

where Σ_1 and Σ_2 are the wetted regions on B_1 and B_2 respectively. The contact angles with the B_i will be $\gamma_i = \arccos(c_i)$. The bilinear form $\mathcal{M}$ must also be modified. If we write σ_i for the curve of contact of Σ with B_i, we have

$$\mathcal{M}(\varphi,\varphi) = \iint_\Sigma |\nabla\varphi|^2 - |S|^2\varphi^2\, d\Sigma + \oint_{\sigma_1} \rho_1\varphi^2\, d\sigma + \oint_{\sigma_2} \rho_2\varphi^2\, d\sigma. \tag{1.9}$$

The boundary conditions for the eigenvalue problem for the operator $\mathcal{L}$ must similarly be adjusted.

2. Comparing bridges between planes and bridges between spheres

In the absence of gravity, a capillary surface is a surface Σ of constant mean curvature which makes a constant contact angle γ with a fixed surface Γ. Suppose that we have such a surface, and, while keeping Σ and its boundary fixed, we replace Γ by a new surface Γ', which still contains $\partial\Sigma$. Suppose this new surface Γ' makes a new constant contact angle γ' with Σ. The general question is how this will effect stability or energy minimality of Σ. At first glance, this question may seem artificial. However, rotationally symmetric liquid bridges between solid spheres are the same surfaces as those between parallel planes. Since much is known about stability of bridges between planes, our hope is that from this knowledge we can infer some information about stability of bridges between spheres.

From (1.2), we can conclude that changing the fixed surface Γ may change the value of ρ, but that the surface integral in $\mathcal{M}$ remains unchanged. It therefore makes sense to compare ρ values for bridges between planes and bridges between spheres. It is known (see [**9**]) that a bridge between parallel planes must be a surface of revolution. (However, there are bridges between spheres which are not surfaces of revolution. See Note 2.1.) Surfaces of revolution having constant mean curvature are called Delaunay surfaces. Their profile curves may be obtained by rolling a conic section along an axis and tracing the path of a focus. Rolling an ellipse results in a curve called an undulary, and the resulting surface is an unduloid. Rolling a hyperbola yields a nodary as a profile curve and a nodoid as the surface. Parabolas give catenaries and catenoids, cylinders come from rolling circles, and spheres come from "rolling" line segments. See [**6**] for more information about Delaunay surfaces.

To make things specific, consider the following situation. Suppose that we have a Delaunay surface generated by a profile passing through the point (x_0, y_0), and that the axis of rotation of the Delaunay surface is the x axis. Suppose that κ_Σ is the curvature of the profile at the point (x_0, y_0) (this agrees with the terminology in (1.3)). The bridge is only part of the Delaunay surface, so let's assume that the bridge lies to the left of the plane $x = x_0$. The profile curves of one case is illustrated in Figure 2, where the center of the sphere is to the right of Γ_o. The other case, where the center is to the left of Γ_o, but the sphere still does not cross the free surface Σ, is in Figure 3. The point of the following calculation is to determine how the value of ρ along the curve of contact will change in going from the Delaunay surface forming a bridge between planes to the Delaunay surface forming a bridge between spheres.

Lemma 2.1. *Suppose that the fixed surface that the bridge Σ contacts is the plane $x = x_0$, whose profile is labeled Γ_o in Figures 2 and 3, and let γ_o be the*

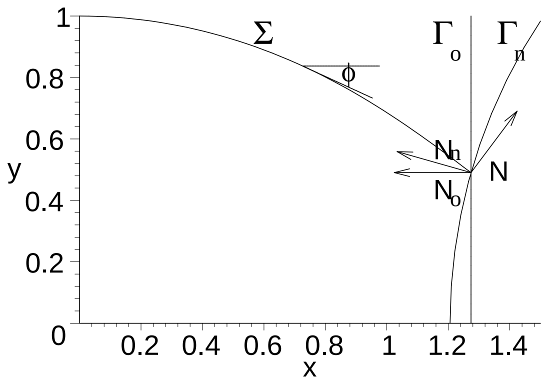

FIGURE 2. Changing the fixed surface, case 1

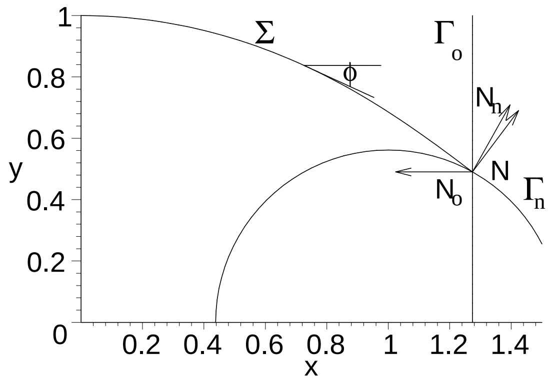

FIGURE 3. Changing the fixed surface, case 2

contact angle, which is between the normals N to Σ, and N_o, to Γ_o. Let ρ_o be the value of ρ for this configuration. Now consider replacing the plane by a sphere going through (x_0, y_0), whose profile is labeled Γ_n in Figures 2 and 3. (The subscript "o" stands for "old", the subscript "n" stands for "new".) Assume that this sphere has a radius of a, with center on the x axis. The contact angle has changed, and the value of ρ has changed to ρ_n. Then

$$\rho_n - \rho_o = \frac{1}{(\cot\eta - \cot\gamma_o)\sin^2\gamma_o}\left(\kappa_\Sigma + \frac{\sin\gamma_o}{y_0}\right), \tag{2.1}$$

where η is the difference between the original contact angle γ_o and the new contact angle γ_n.

Proof. Case 1: We have that ρ_o is

$$\rho_o = \kappa_\Sigma \cot\gamma_o,$$

since the curvature of the fixed surface is zero. Now replace the plane by Γ_n. The contact angle is now the angle between N and N_n, and has changed to $\gamma_n = \gamma_o - \eta$, where

$$\eta = \arcsin\left(\frac{y_0}{a}\right).$$

We therefore have that the new value of ρ is

$$\rho_n = \kappa_\Sigma \cot\left(\gamma_o - \eta\right) + \frac{1}{a}\csc\left(\gamma_o - \eta\right),$$

since the sectional curvature of the fixed surface has decreased from 0 to $-1/a$. Trigonometric identities for $\cot(A-B)$ and $\csc(A-B)$ give

$$\begin{aligned}
\rho_n - \rho_o &= \kappa_\Sigma\left(\frac{\cot\gamma_o\cot\eta + 1}{\cot\eta - \cot\gamma_o} - \cot\gamma_o\right) + \frac{1}{a\sin\gamma_o\sin\eta}\left(\frac{1}{\cot\eta - \cot\gamma_o}\right)\\
&= \kappa_\Sigma\left(\frac{\cot\gamma_o\cot\eta + 1 - \cot\gamma_o\cot\eta + \cot^2\gamma_o}{\cot\eta - \cot\gamma_o}\right)\\
&\quad + \frac{1}{a\sin\gamma_o\sin\eta}\left(\frac{1}{\cot\eta - \cot\gamma_o}\right)\\
&= \frac{1}{\cot\eta - \cot\gamma_o}\left(\kappa_\Sigma\csc^2\gamma_o + \frac{1}{a\sin\gamma_o\sin\eta}\right)\\
&= \frac{1}{(\cot\eta - \cot\gamma_o)\sin^2\gamma_o}\left(\kappa_\Sigma + \frac{\sin\gamma_o}{y_0}\right).
\end{aligned}$$

as desired.

The calculation for case 2 is similar, except that now $\eta = \frac{\pi}{2} - \arcsin\left(\frac{y_0}{a}\right)$, and is omitted. ∎

Now, suppose that we have a bridge with a convex profile. In both case 1 and case 2, one can show that $0 < \eta < \gamma_o < \pi$, so that $\cot\eta - \cot\gamma_o > 0$. Therefore, the sign of $\kappa_\Sigma + \frac{\sin\gamma_o}{y_0}$ will determine whether the value of ρ has increased or decreased. From this we will be able to determine stability of convex bridges between spheres. We first need to recall some facts about the profiles of Delaunay surfaces.

If $(x(s), y(s))$ is an arclength parametrization of the profile of a Delaunay surface, with inclination angle $\varphi(s)$, (see Figure 2 for φ) and with mean curvature H, then we have the following system of ordinary differential equations:

(2.2a) $$\frac{dx}{ds} = \cos\varphi,$$

(2.2b) $$\frac{dy}{ds} = \sin\varphi, \text{ and}$$

(2.2c) $$\frac{d\varphi}{ds} = \frac{\cos\varphi}{y} + 2H,$$

(see [**10**]). From this system, it's easy to see that

$$\frac{d}{ds}\left(y\cos\varphi + Hy^2\right) = 0,$$

so that $y\cos\varphi + Hy^2$ is constant along Delaunay profiles. The value of this constant has a geometric meaning.

Lemma 2.2. *Let the constant value of* $y\cos\varphi + Hy^2$ *on the profile of a Delaunay surface be called* c. *If* $Hc > 0$, *then the profile is a nodary, and if* $Hc < 0$ *then the profile is an undulary.*

Proof. This is already known (see [**7**], e.g.), but I was not able to locate a proof in the literature, and it is not hard to present one. It is easy to check that $c = 0$ for a sphere, so that this case will not occur. Substitute the definition of c into (2.2c) to see that

$$\frac{d\varphi}{ds} = H + \frac{c}{y^2}.$$

If H and c have the same signs, then $\varphi(s)$ is monotone on the profile. This rules out undularies, and a catenary is not possible for $H \neq 0$, hence we must have a nodary. On the other hand, suppose that H and c have different signs. From the definition of c it is clear that $\varphi = \frac{\pi}{2}$ cannot be on the profile. The only possibility in this case is an undulary (we consider a circular cylinder as a special case of an undulary). ∎

Lemma 2.3. *Suppose that we have a rotationally symmetric bridge* Σ *with a convex profile contacting a plane as in Figure* 2 *or* 3. *Suppose that we replace the plane* Γ_o *with a sphere* Γ_n *as in the figure. If* Σ *is a portion of an unduloid, then* $\rho_n > \rho_o$, *and if* Σ *is a portion of a nodoid, then* $\rho_n < \rho_o$. *In particular, if we take a convex bridge between parallel planes and replace the planes by spheres, both values of* ρ *in* (1.9) *will increase if* Σ *is a portion of an unduloid, and decrease if* Σ *is a portion of a nodoid.*

Proof. As noted before, the sign of the change of ρ is the same as the sign of $\kappa_\Sigma + \frac{\sin\gamma_o}{y_0}$. But this last quantity will be equal to

$$\begin{aligned}\frac{d\varphi}{ds} + \frac{\cos\varphi_o}{y_0} &= 2\left(\frac{\cos\varphi_o}{y_0} + H\right)\\ &= \frac{2}{y_0^2}\left(y_0\cos\varphi_o + Hy_0^2\right),\end{aligned}$$

where φ_o is the inclination angle of the profile at the right endpoint, so $\varphi_o = \frac{\pi}{2} - \gamma_o$. So, we have

$$\kappa_\Sigma + \frac{\sin\gamma_o}{y_0} = 2\frac{c}{y_0^2},$$

where c has the same meaning as in Lemma 2.2. From (2.2c), it is clear that for a convex profile we have $H < 0$, so $c > 0$ for an undulary and $c < 0$ for a nodary, concluding the proof. ∎

Theorem 2.1. *Suppose that Σ is a rotationally symmetric bridge between spheres, whose profile is given as a solution to (2.2), and that $\frac{d\varphi}{ds} < 0$ and $\frac{dx}{ds} > 0$ on the bridge profile including the endpoints. If Σ is a section of a nodoid, it is unstable, and if Σ is a section of an unduloid or a sphere, then Σ is stable, and in fact a local constrained energy minimum. (We do not assume that the spheres have equal radius or that the contact angles are equal.)*

Proof. It is known that for bridges between parallel planes, a convex bridge is a constrained local energy minimum modulo translations in directions parallel to the planes (see [**13**], [**10**]). In the proof in [**13**], the quadratic form

$$\mathcal{M}_o(\varphi,\varphi) = \iint_\Sigma |\nabla\varphi|^2 - |S|^2\varphi^2\, d\Sigma + \oint_{\sigma_1} \rho_{o,1}\varphi^2\, d\sigma + \oint_{\sigma_2} \rho_{o,2}\varphi^2\, d\sigma \tag{2.3}$$

was considered. We write $\rho_{o,i}$ for the "old" value of ρ_i as in Lemma 2.1. It was shown that this is strongly positive (i.e., that there is an $\varepsilon > 0$ so that $\mathcal{M}_o(\varphi,\varphi) \geq \varepsilon\,\|\varphi\|^2$, where $\|\cdot\|$ is the $H^1(\Sigma)$ norm) on the subspace of $1^\perp$ of φ's which are also orthogonal in $H^1(\Sigma)$ to infinitesimal translations parallel to the fixed planes. This strong positivity leads directly to the statement about energy minimality. However, if μ corresponds to a translation parallel to the fixed planes, we must have $\mathcal{M}_o(\mu,\mu) = 0$, since $\mathcal{M}$ is the second Fréchet derivative of energy, and energy is unchanged by translations. In fact, the eigenvalue problem (1.5), (1.6) will have a single negative eigenvalue, 0 as an eigenvalue of multiplicity two, and all other eigenvalues positive. Using the same notation as in [**13**], we let μ_1 and μ_2 span the subspace of infinitesimal translations parallel to the fixed planes. With the parametrization of Σ given in section 5 of that paper, we have

$$\mu_1(u,v) = \frac{\cos v}{\sqrt{1+(f')^2}}$$

and

$$\mu_2(u,v) = \frac{\sin v}{\sqrt{1+(f')^2}}$$

(the profile is given as the graph of $r = f(u)$). These functions also span the kernel of the eigenvalue problem (1.5), (1.6).

If $\rho_{n,i}$, the values of ρ for the new configuration (i.e., when the planes are replaced by spheres) satisfy $\rho_{n,1} < \rho_{o,1}$ and $\rho_{n,2} < \rho_{o,2}$, then the bridge is unstable (and hence not a constrained local energy minimum) for the new configuration of fixed surfaces. The reason is simple: we must have $\mathcal{M}_n(\varphi,\varphi) < \mathcal{M}_o(\varphi,\varphi)$ for any φ which is non-zero on a set of positive measure on the boundary of Σ. Therefore, in particular we must have $\mathcal{M}_n(\mu_1, \mu_1) < 0$. But translations in the original configuration also conserve volume, so $\iint_\Sigma \mu_1 \, d\Sigma = 0$, i.e., $\mu_1 \in 1^\perp$. The second variation of energy is negative for this infinitesimally volume conserving perturbation, so we have instability in the case that $\rho_{n,i} < \rho_{o,i}$. From Lemma 2.3, we therefore have instability when the bridge is a portion of a nodoid.

If $\rho_{n,i} > \rho_{o,i}$, i.e., when the bridge is a portion of an unduloid, we expect the new configuration to be more stable (in some sense) than the old one. In fact, we will see that in this case $\mathcal{M}_n$ is strongly positive on all of $1^\perp$. Indeed, suppose that this is not the case. We certainly know that $\mathcal{M}_n$ is non-negative on this space, since M_o is non-negative on this space and $\mathcal{M}_n(\varphi, \varphi) \geq \mathcal{M}_o(\varphi, \varphi)$. So, if M_n is not strongly positive on $1^\perp$, there must exist a sequence $\{\varphi_k\}$ in $1^\perp$ for which $\|\varphi_k\| = 1$ and $\lim_{k\to\infty} \mathcal{M}_n(\varphi_k, \varphi_k) = 0$.

Projecting this sequence onto the span of μ_1 and μ_2, we write

$$\varphi_k = a_k \mu_1 + b_k \mu_2 + \varphi_k^*.$$

Note that since $\iint_\Sigma \mu_i \, d\Sigma = 0$, we have $\varphi_k^* \in 1^\perp$. By going to a subsequence, we may assume that $\{a_k\}$ and $\{b_k\}$ converge to a and b, respectively. Now,

$$\begin{aligned}
\mathcal{M}_n(\varphi_k, \varphi_k) &\geq \mathcal{M}_o(\varphi_k, \varphi_k) \\
&= \mathcal{M}_o(a_k\mu_1 + b_k\mu_2 + \varphi_k^*, a_k\mu_1 + b_k\mu_2 + \varphi_k^*) \\
&= \mathcal{M}_o(a_k\mu_1 + b_k\mu_2, a_k\mu_1 + b_k\mu_2) \\
&\quad + 2\mathcal{M}_o(a_k\mu_1 + b_k\mu_2, \varphi_k^*) + \mathcal{M}_o(\varphi_k^*, \varphi_k^*) \\
&= \mathcal{M}_o(\varphi_k^*, \varphi_k^*) \geq \varepsilon \|\varphi_k^*\|^2, \qquad (2.4)
\end{aligned}$$

where the terms $\mathcal{M}_o(a_k\mu_1 + b_k\mu_2, a_k\mu_1 + b_k\mu_2)$ and $\mathcal{M}_o(a_k\mu_1 + b_k\mu_2, \varphi_k^*)$ vanish by (1.7) and the fact that $\mathcal{L}(\mu_i) = 0$ on Σ, $\mathbf{b}(\mu_i) = 0$ on σ.

From (2.4) and the fact that $\lim_{k\to\infty} \mathcal{M}_n(\varphi_k, \varphi_k) = 0$, we conclude that $\{\varphi_k^*\}$ goes to zero in $H^1(\Sigma)$, thus

$$\lim_{k\to\infty} \varphi_k = a\mu_1 + b\mu_2$$

in $H^1(\Sigma)$. An immediate consequence is that a and b cannot both be zero, since all of the φ_k's have length 1 in $H^1(\Sigma)$. This leads to a contradiction. Since $M_n(\varphi, \varphi)$ is continuous on $H^1(\Sigma)$,

$$\mathcal{M}_n(a\mu_1 + b\mu_2, a\mu_1 + b\mu_2) = \lim_{k\to\infty} \mathcal{M}_n(\varphi_k, \varphi_k) = 0.$$

However, $a\mu_1 + b\mu_2$ is not identically zero on $\partial\Sigma$. The reason is that it represents the component normal to Σ of a non-trivial translation parallel to the original fixed planes. Therefore

$$\mathcal{M}_n(a\mu_1 + b\mu_2, a\mu_1 + b\mu_2) > \mathcal{M}_o(a\mu_1 + b\mu_2, a\mu_1 + b\mu_2) = 0,$$

a contradiction. Thus $\mathcal{M}_n$ is strongly positive on all of $1^\perp$, proving that a bridge between spheres which is convex and part of an unduloid must be a local energy minimum. ▮

Note 2.1. *No claim about energy minimality was made in the case that the bridge is a section of a sphere. In this case, the spectrum of the eigenvalue problem* (1.5), (1.6) *remains the same as in the problem of a bridge between parallel planes, so that 0 is an eigenvalue of multiplicity two. What is happening at the symmetrically placed spherical bridge is that there is a "wine cup" bifurcation. By shooting arguments, one can show that this spherical bridge is embedded in a family of Delaunay surfaces which form bridges between the balls. But by simple trigonometric arguments, one can also construct a family of asymmetrically placed spherical bridges (see Figure* 4*). For every volume larger than V_0, that of the symmetrically placed spherical bridge, there is a one-parameter family of asymmetric spherical bridges, all of which rotate into each other. As volume decreases to V_0, these all collapse to the symmetrically placed spherical bridge, so that the symmetrically placed spherical bridge is a limiting member of this family as well.*

Note 2.2. *A cylindrical bridge between spheres is a limiting case of unduloids. Conditions under which the cylinder is a local energy minimum are derived in* [**11**].

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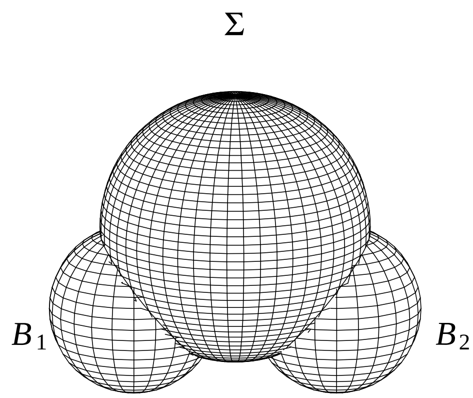

FIGURE 4. Asymmetrically placed spherical bridge

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