PROCEEDINGS OF THE FIFTH INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
June 16 – 19, 2004, Pomona, CA, USA pp. **1–6**

COMMENTS ON RADIALLY SYMMETRIC LIQUID BRIDGES WITH INFLECTED PROFILES

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Abstract. A geometrical argument is outlined to show the (already known) instability of an inflected liquid bridge between parallel planes in the case of equal contact angles. In contrast to the behavior of liquid bridges between parallel planes, it is shown that a liquid bridge between spheres exists which is stable and has two inflections. Along the way, a result relating stability and dH/dV for a family of capillary surfaces is established.

1. **Introduction.** The stability of a liquid bridge between parallel planes has been studied in a number of papers (e.g., [1], [6], [10], [11], [18]). That of a liquid bridge between solid balls is also physically important, but has not received as much attention (see [2], [14], however). One might hope that some of the analysis goes over from the first problem to the second. The purpose of this paper is to sound a cautionary note. One result from the study of bridges between parallel planes is that no such bridge can have an inflection point in its profile and be stable. This was derived in [11], but from the proof it is not obvious how the result depends on the geometry of the problem. In section 2, a geometric approach is outlined which makes clear why one would expect the appearance of an inflection to lead to instability, at least for the case of parallel planes. Section 3 establishes a general result about capillary surfaces, relating stability and the sign of dH/dV in a family of surfaces with one negative eigenvalue for a certain eigenvalue problem. In section 4, the result of the previous section is used to give an example of a stable bridge between two spheres which has two inflections. This shows that the geometrical reasoning in section 2 is quite specific to parallel planes.

2. **Parallel planes.** In [11] (see also [16]) it was shown that a stationary bridge between planes is stable if 1) a certain Sturm-Liouville problem (equation 1.2 of [11]) has precisely one negative eigenvalue and 2) the stationary bridge may be embedded in a smoothly parameterized family of stationary bridges for which dH/dV has the correct sign. Here V is volume and H is mean curvature. For the case of equal contact angles, the failure of the first condition coincides with the appearance of an inflection on the boundary ([6]). As λ_1 passes through zero a pitchfork bifurcation occurs. This is not mysterious; there is a simple geometric reason. (Essentially this argument was used in [12] to compute behavior of bridges near pitchfork bifurcations.)

We will assume that the profile curve is $f(x)$, and that the parallel planes are $x = 0$ and $x = 1$. The inclination angle ϕ is the angle of the curve with the

1991 *Mathematics Subject Classification.* Primary: 53A10, 49Q10.
Key words and phrases. capillarity, stability, inflections.

horizontal. If a rotationally symmetric liquid bridge meets the planes with a contact angle γ, then at $x = 0$ the inclination is $\frac{\pi}{2} - \gamma$ and at $x = 1$ the inclination is $\gamma - \frac{\pi}{2}$. Call the point $(0, f(0))$ P and the point $(1, f(1))$ Q.

Theorem 1. *For liquid bridges between parallel planes and with equal contact angles $\gamma \neq \frac{\pi}{2}$, the profile with second derivative vanishing on the boundary represents a bifurcation between a family of zero-inflection profiles, a family of two-inflection profiles, and two families of one-inflection profiles.*

Proof: Rotationally symmetric surfaces of constant mean curvature are called Delaunay surfaces, and are obtained by rolling a conic along a fixed line, tracing the focus, and revolving the resulting curve around the line. They are classified as: unduloids, obtained by rolling ellipses, nodoids, obtained by rolling hyperbolas, catenoids, obtained from parabolas, cylinders, obtained by rolling circles, and spheres, which may be considered as limiting cases of either nodoids or unduloids. (See, e.g., [7].) Following [7], we will call the profile curve of an unduloid an undulary and the profile curve of a nodoid a nodary. Since we consider a profile with an inflection, it must be a portion of an undulary, since profile of Delaunay surfaces which aren't unduloids don't have inflections. We will consider the profile as a function of x, so that the inclination angle is the angle with the horizontal. The inflections are local extremes for the inclination angle: The inclination angle ϕ of the unduloid will vary between $-\left|\gamma - \frac{\pi}{2}\right|$ and $\left|\gamma - \frac{\pi}{2}\right|$, with each of these occurring at an endpoint of the interval (i.e., on the planes). From formulas in [4], for any unduloid the sine of the inclination angle goes from $-\frac{r_b - r_a}{r_b + r_a}$ to $\frac{r_b - r_a}{r_b + r_a}$, where r_b is the maximum radius of the unduloid and r_a is the minimum radius of the unduloid. It's not hard to show from this that $\sin\phi$ varies from $-e$ to e, where e is the eccentricity of the generating ellipse.

Start with the original profile, and call the eccentricity of its generating ellipse e_0. Increase the eccentricity of the generating ellipse slightly, say by keeping the length of the major axis the same but decreasing the length of the minor axis slightly. By the above comments, the maximum inclination angle of the resulting unduloid will be slightly larger than $\left|\gamma - \frac{\pi}{2}\right|$ and the minimum inclination angle will be slightly smaller than $-\left|\gamma - \frac{\pi}{2}\right|$. From this it follows that instead of the single point P on the original unduloid with inclination $\frac{\pi}{2} - \gamma$ at $x = 0$, this point will split to be a pair of points close to $x = 0$ with that inclination, call them P_1 and P_2, with an inflection in between these points. Similarly, the point Q with inclination $\gamma - \frac{\pi}{2}$ splits into a pair of points Q_1, Q_2. From this slight perturbation of the original unduloid we can construct four bridges with the correct contact angles. Simply take the perturbed unduloid, cut it at P_i on the left and Q_j on the right, i,$j = 1$, 2, and scale the resulting bridge so that the difference of the x coordinates of P_i and Q_j is 1. It's not hard to see that these four bridges all approach the original bridge uniformly as the eccentricity decreases to e_0, hence the original bridge is a bifurcation point. As for the number of inflections: there will either be 0, 1, or 2 inflections depending on the choice of the P_i and the Q_j.

3. **General results on stability.** We first derive some general results relating to the stability criteria in [15], in particular case 3 of Theorem 2.2 of that paper. Specifically, suppose that Σ is a capillary surface so that the eigenvalue problem

$$\mathcal{L}(\psi) \equiv -\Delta\psi - |S|^2\,\psi = \lambda\psi \tag{1}$$

on Σ and

$$\mathbf{b}(\psi) \equiv \psi_1 + \rho\psi = 0 \tag{2}$$

on $\partial\Sigma$ (see [15] for the definitions of the coefficients) satisfies $\lambda_0 < 0 < \lambda_1$. Let ζ be the solution to

$$\mathcal{L}(\zeta) = 1 \tag{3}$$

on Σ and

$$\mathbf{b}(\zeta) = 0, \tag{4}$$

which is involved in case 3 of Theorem 2.2. The criterion is that if $\iint_\Sigma \zeta < 0$ then Σ is a constrained local minimum for energy, and if $\iint_\Sigma \zeta > 0$ then Σ is unstable. There is an important interpretation of ζ as determined in (3), (4). This comes from the following theorem.

Theorem 2. *In the curvilinear coordinates set-up of* [15], *suppose that Σ has constant mean curvature H_0 with constant contact angle γ. (The mean curvature is with respect to the unit normal $\vec{N}$ to Σ which points out of the liquid. For example, if the drop is a ball, the mean curvature is negative.) Suppose that $\phi(p,t)$, $p \in \Sigma$, $t \in (-a,a)$ is a smoothly parameterized collection of functions on Σ so that for each t, the graph of $\phi(p,t)$ in curvilinear coordinates has constant mean curvature $H(t)$ and constant contact angle γ, and $\phi(p,0) = 0$. Then $\phi_t(p,0)$ satisfies*

$$\mathcal{L}(\phi_t(p,0)) = -2H'(0) \tag{5}$$

on Σ, and

$$\mathbf{b}(\phi_t(p,0)) = 0 \tag{6}$$

on $\partial\Sigma$.

Proof: As in [15], let $\vec{x}(p,w)$ be curvilinear coordinates near Σ, with $x(p,0)$ being Σ itself, and $\vec{x}_w(p,0) \cdot \vec{N} = 1$, where $\vec{N}$ is the normal to Σ. Then the graph of $\phi(p,t)$ in curvilinear coordinates is the surface $\Sigma(t)$, parameterized by $p \in \Sigma$ as $\vec{x}(p,\phi(p,t))$. We want to differentiate

$$\mathcal{M}(\vec{x}(p,\phi(p,t))) = H(t) \tag{7}$$

with respect to t, where $\mathcal{M}$ denotes mean curvature. We wish to apply formula (4.1b) of [9], but that formula is for normal variations, and the variation vector field

$$\frac{\partial}{\partial t}\vec{x}(p,\phi(p,t)) = \vec{x}_w(p,\phi(p,t))\phi_t$$

need not be normal to Σ at $t = 0$. However, the normal component is $\phi_t(p,0)$ (by assumption on $\vec{x}$), and the tangential component may be neglected. The argument for neglecting the tangential component is the same as that which appears in the proof of Lemma 3.7 of [13], that the tangential component may be viewed as the derivative of a mapping of Σ into itself, and mean curvature is constant for such a map. This establishes (5). The boundary condition (6) is established as in [17].

Corollary 1. *Suppose that $\Sigma(t)$ is a smoothly parameterized family of surfaces, with mean curvature constantly $H(t)$ on each surface, and with each surface making a constant contact angle γ with a smooth fixed surface Λ. Let $V(t)$ be the volume of the drop bounded by $\Sigma(t)$. Suppose that, for $\Sigma(0)$, the eigenvalues for (1), (2) satisfy $\lambda_0 < 0 < \lambda_1$. If $H'(0)V'(0) > 0$, then $\Sigma(0)$ is stable (in fact, a local energy minimum) and if $H'(0)V'(0) < 0$ then $\Sigma(0)$ is unstable.*

Proof: First, since 0 is not an eigenvalue, (5) implies that $H'(0) \neq 0$. Now, define $\phi(x,t)$ so that $\Sigma(t)$ is the graph of $\phi(x,t)$ in suitable curvilinear coordinates over $\Sigma(0)$. Then ζ, given in (3) and (4), is $-\frac{1}{2H'(0)}\phi_t(x,0)$. $V'(0)$ is

$$V'(0) = \iint_{\Sigma(0)} \phi_t(x,0) = -2H'(0) \iint_{\Sigma(0)} \zeta.$$

Thus, if $V'(0)$ and $H'(0)$ have the same signs, the criterion mentioned at the start of this section gives that $\Sigma(0)$ is a constrained local minimum for energy (and a fortiriori stable). If $V'(0)$ and $H'(0)$ differ in sign, then instability follows.

Remark 1. This generalization of a result in [10] (already mentioned in Section 2) is not surprising, in view of [8]. However, the situation is not quite the same as that considered in [8] since surface area is not Fréchet differentiable in the Hilbert spaces one would prefer to use.

4. **A stable bridge between spheres with two inflections.** In [14], the stability of a cylindrical liquid bridge between solid balls was analyzed. A variation of this leads to a stable bridge between two solid balls whose profile has two inflection points. The relevant result (Theorem 2.2) is that a cylinder Σ of radius 1 and height h forming a bridge between spheres of radius $a > 1$ is a local energy minimum if $h < 2\pi - 2\arctan\left(\sqrt{a^2-1}\right)$, and in fact (using the terminology of section 3) satisfies $\lambda_0 < 0 < \lambda_1$ and $\iint_\Sigma \zeta < 0$. Of course, to have this cylinder be stationary, we need the contact angle γ to be $\arccos(1/a)$.

Lemma 1. *In the case described above, the cylinder is not a bifurcation point for the collection of solutions with varying volume, but fixed contact angle, to the problem of a liquid bridge between spheres. This should be understood in the setting of* [15]*, where the bridges are considered as graphs in curvilinear coordinates, and the spheres are fixed.*

Proof: This is almost immediate, but to apply the implicit function theorem as in the introduction to [3], the actual operator that we need to have invertible is the operator A defined in [15]. Since the eigenvalues of (1), (2) are bounded from zero, the invertibility of A follows from Lemma 2.8 of that paper.

Theorem 3. *There exists a stable liquid bridge between balls of equal radius which is radially symmetric and has two inflection points in its profile. In fact, this liquid bridge is a local energy minimum (of course, subject to the volume constraint).*

Proof: Take a specific cylindrical liquid bridge of radius 1, between two balls of some radius $a > 1$, and with height h satisfying $\pi < h < 2\pi - 2\arctan\left(\sqrt{a^2-1}\right)$. Since this cylinder does not represent a bifurcation, it is embedded in a smoothly parameterized family of stationary liquid bridges, where we may parameterize by the volume that the bridge contains. Call the surfaces of this family $\Sigma(V)$, and let V_0 be the volume of the cylindrical bridge, so that the cylinder is $\Sigma(V_0)$, and the range of volumes for which the parametrization is good is at least $(V_0-\varepsilon, V_0+\varepsilon)$, for some $\varepsilon > 0$.

Since this family is parameterized by volume, we may deduce some facts about it. In particular, each bridge in $\Sigma(V)$, $V \in (V_0-\varepsilon, V_0+\varepsilon)$ must be rotationally symmetric. Indeed, if this were not so, we could revolve a bridge to obtain a continuum of bridges containing the same volume. For much the same reason, each bridge in $\Sigma(V)$ is symmetric across the plane which is the perpendicular bisector of the line segment between the centers of the spheres. Since the bridges are rotationally symmetric, they must be Delaunay surfaces. In particular, since a cylinder is a special case of an unduloid, the family $\Sigma(V)$ consists of unduloids, at least for V sufficiently close to V_0.

In addition, for values of V sufficiently close to V_0, we will see that $\Sigma(V)$ represents bridges which are stable, and in fact local energy minima. Since the surfaces depend differentiably on V, the coefficients of the eigenvalue equations (1), (2) depend differentiably on V as well. Certainly, therefore, the values of the eigenvalues depend continuously on V (see [5], Chapter 6). Thus for V sufficiently close to V_0, the bridge $\Sigma(V)$ will have precisely one negative eigenvalue, with the rest strictly positive. From now on, we will assume that V is close enough to V_0 for this to occur. We may therefore apply the stability criterion mentioned in section 3.

Since $\iint_\Sigma \zeta < 0$, it follows from Theorem 2 and Corollary 1 that $\frac{dH}{dV} < 0$ at $V = V_0$. Since the bridges are smoothly parameterized by V, it follows that for V sufficiently close to V_0 we still have $\frac{dH}{dV} < 0$. Thus, using Corollary 1 we conclude that that for all V sufficiently close to V_0 there are stable unduloids (in fact, they are constrained local energy minima) which are arbitrarily close to the stable cylinder.

The last thing to do is to show that unduloids which are close enough to the stable cylinder will have two inflections. Since the cylinder has radius 1, one can think of its profile as generated by rolling a circle of arc length 2π and tracing the center. An unduloid which is uniformly close to this cylinder is generated by rolling an ellipse of small eccentricity and of arc length close to 2π. Take c to be half the distance between the foci of the ellipse. By the asymptotic formulas in the appendix of [11], we have that the undulary obtained by rolling an ellipse of arc length equal to 2π is

$$f(x;c) = 1 + c\cos x + \frac{c^2}{2}\left(1 - \frac{1}{2}\cos 2x\right) + O\left(c^3\right).$$

Since the arc length of the ellipse is close to, but not necessarily equal to 2π, we introduce a scaling factor L and consider

$$g(x;c,L) = Lf\left(\frac{x}{L};c\right).$$

The aim is to show that as $(c,L) \to (0,1)$, the distance between successive inflection points of g tends to π. Since any unduloid close to the stable cylinder will have (c,L) close to $(0,1)$, this will be enough to prove the result.

We seek roots of g_{xx}, and attempt to see how the location of these roots changes as (c,L) tends to $(0,1)$. Looking at g_{xx}/c instead, to avoid problems at $c=0$, we have

$$G(x;c,L) \equiv \frac{g_{xx}(x;c,L)}{c} = -\frac{1}{L}\cos\left(\frac{x}{L}\right) + \frac{c}{L}\cos\left(\frac{2x}{L}\right) + O\left(c^2\right).$$

At $(c,L) = (0,1)$ there are neighboring roots at $-\pi/2$ and $\pi/2$. Let $x_1(c,L)$ be the root of G near $-\pi/2$ and $x_2(c,L)$ the root near $\pi/2$. It is an exercise in using the Implicit Function theorem to show that

$$G(x_i(c,L);c,L) = 0$$

defines $x_i(c,L)$ as a differentiable function of c and L in a neighborhood of $(0,1)$. In particular, we may conclude that

$$\lim_{(c,L)\to(0,1)} x_2(c,L) - x_1(c,L) = \pi.$$

Thus, unduloids near the stable cylinder will have profiles with inflections separated by a distance close to π. Since the stable cylinder we're looking at was chosen to have h larger than π, the nearby stable unduloids will have at least one inflection point in their profiles. However, we've observed that since there is no bifurcation nearby, the stable unduloids must be symmetric across the plane which is the perpendicular bisector of the line segment joining the centers of the fixed spheres. Thus the stable unduloids near the stable cylinder must have at least two inflections in their profiles, concluding the proof.

Acknowledgements. I'd like to thank Robert Finn for organizing the enjoyable and fascinating special session on capillarity, which served as an inspiration for this paper.

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Received September 14, 2004.

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