Inverse Obstacle Scattering: Local uniqueness and iterative methods in reconstruction

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Joint work with:

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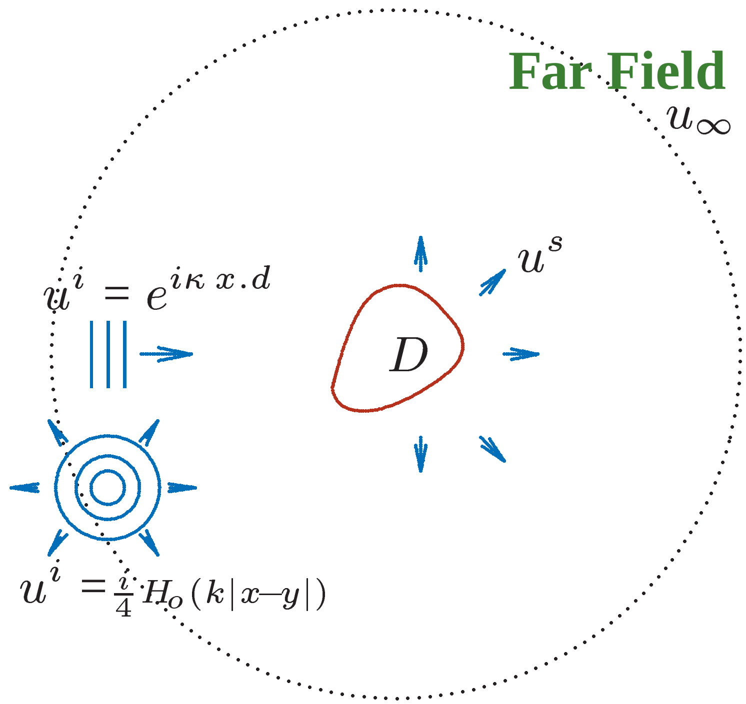
Program Script managed in **C**: The Kernighan-Ritchie version.
Set in plain TEX: No pretentious, aftermarket "improvements".

Inverse Obstacle Scattering

A time-harmonic acoustic or electromagnetic plane wave $u^i = e^{i\kappa\, x.d}$ or point source $\frac{i}{4}\mathrm{H_o}(\mathrm{k}|\mathrm{x}-\mathrm{y}|)$ is fired at a cylindrical obstacle D of unknown shape and location.

The wave u^s scattered from this object is measured at "infinity" – the far field pattern u_∞.



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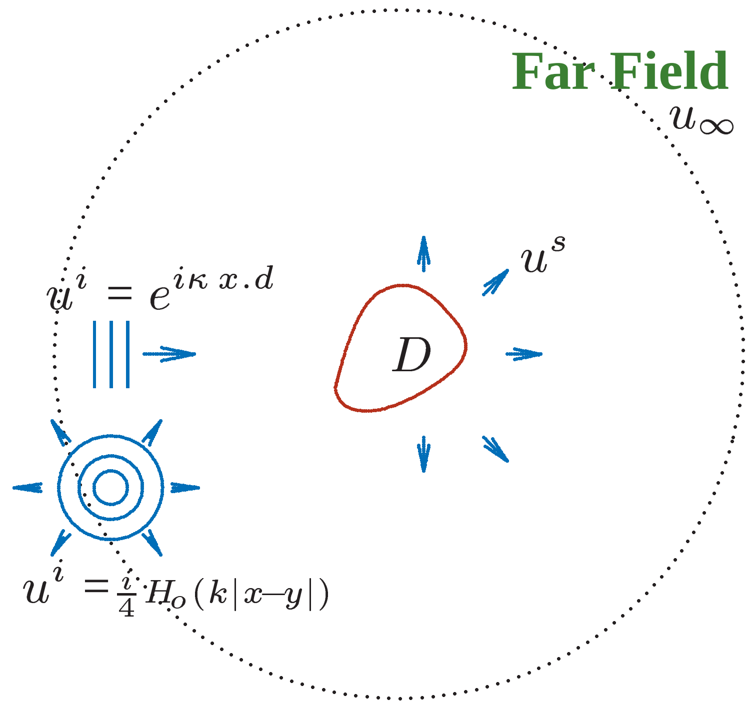

The total wave $u = u^i + u^s$ is modeled by an exterior boundary value problem for the Helmholtz equation:

$$\triangle u + \kappa^2 u = 0 \qquad \text{in } \mathbb{R}^n \setminus \bar{D}$$

with positive wave number κ and a boundary condition on ∂D:

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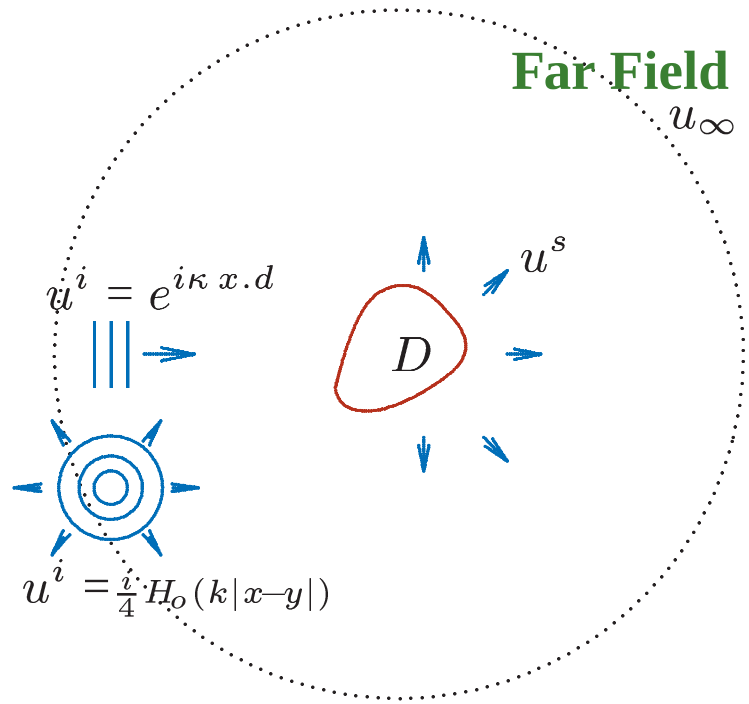

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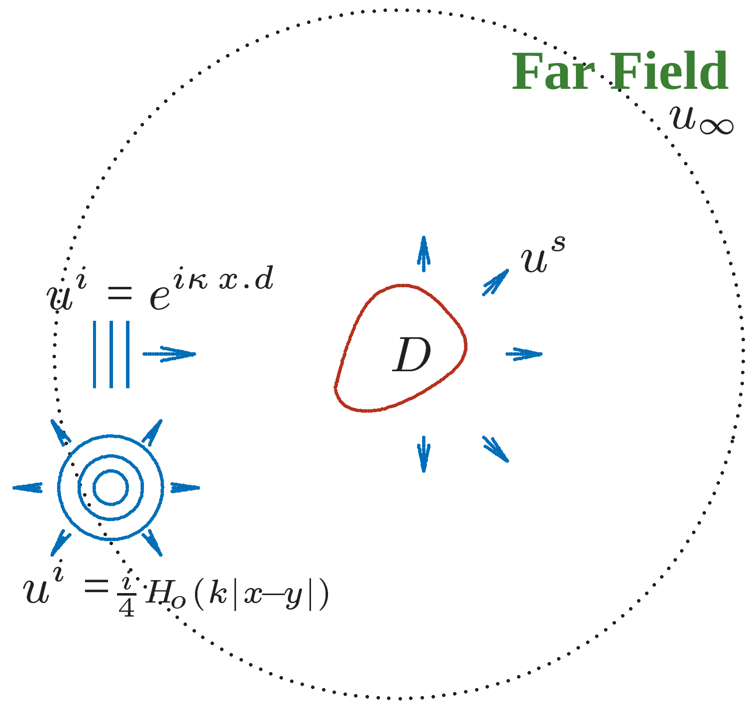

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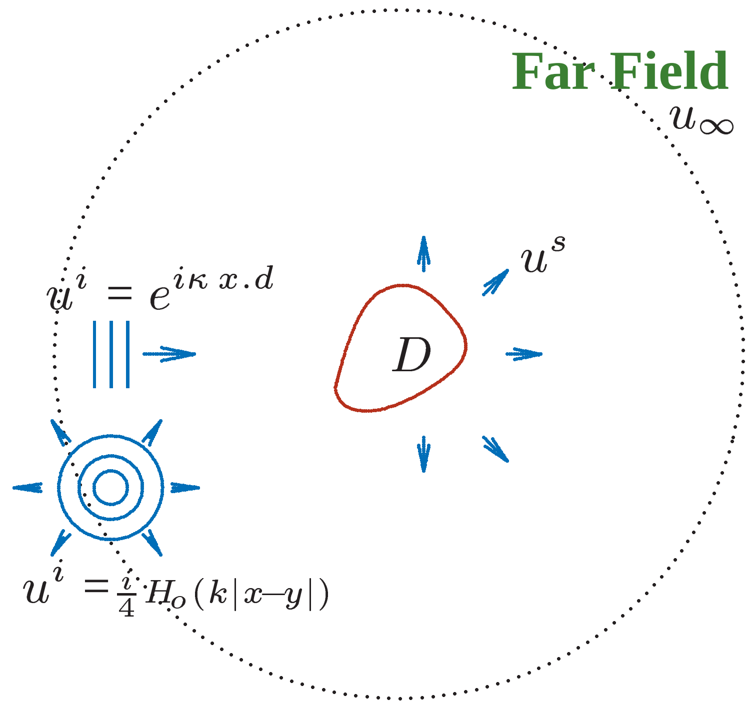

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$$\frac{\partial u}{\partial \nu} + i\kappa\lambda u = 0 \qquad \text{impedance condition}$$

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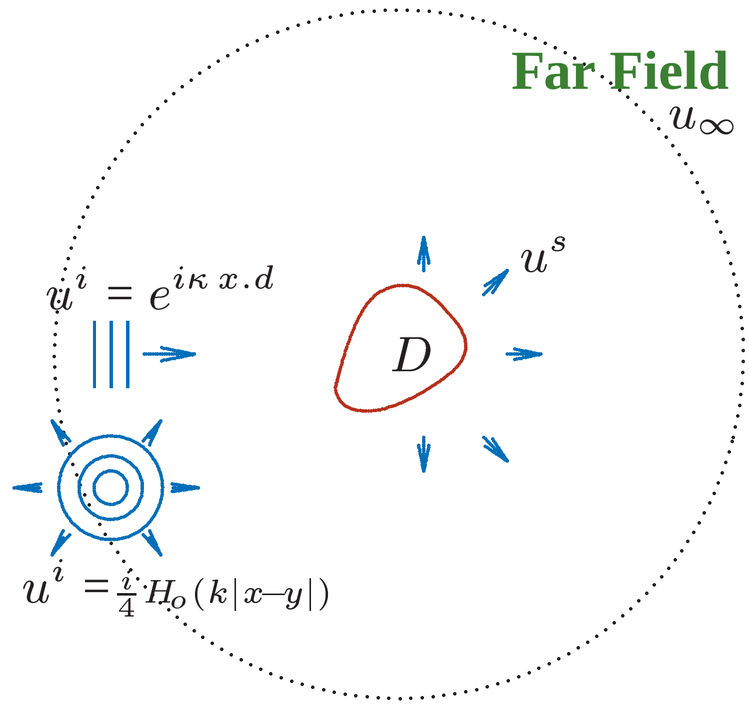

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$$\lfloor u \rfloor_{\partial D} = \lfloor \mu \frac{\partial u}{\partial \nu} \rfloor_{\partial D} = 0 \qquad \text{transmission condition}$$

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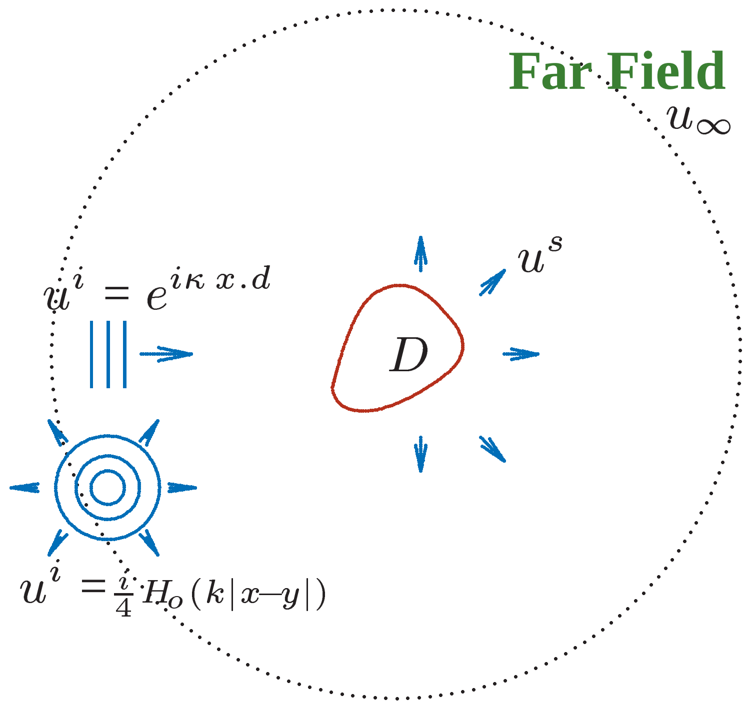

The scattered wave u^s is required to satisfy the Sommerfeld radiation condition uniformly in all directions $\hat{x} = x/|x|$

$$\frac{\partial u^s}{\partial r} - i\kappa u^s = o\left(\frac{1}{\sqrt{r}}\right), \quad r = |x| \to \infty, .$$

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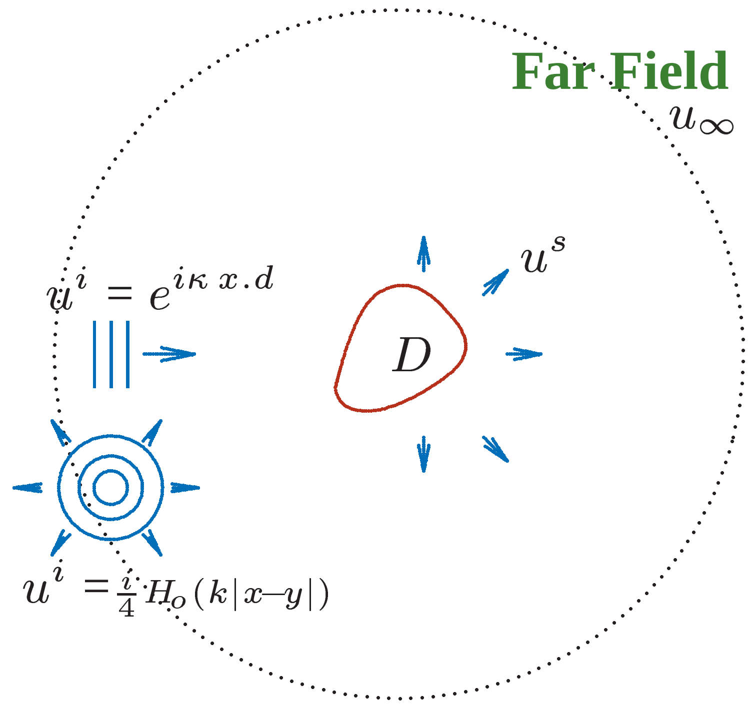

This implies the asymptotic behaviour

$$u^s(x) = \frac{e^{i\kappa x}}{|x|^{(n-1)/2}} \left(u_\infty(\hat{x}; d) + O\left(\frac{1}{|x|}\right)\right), \quad |x| \to \infty, .$$

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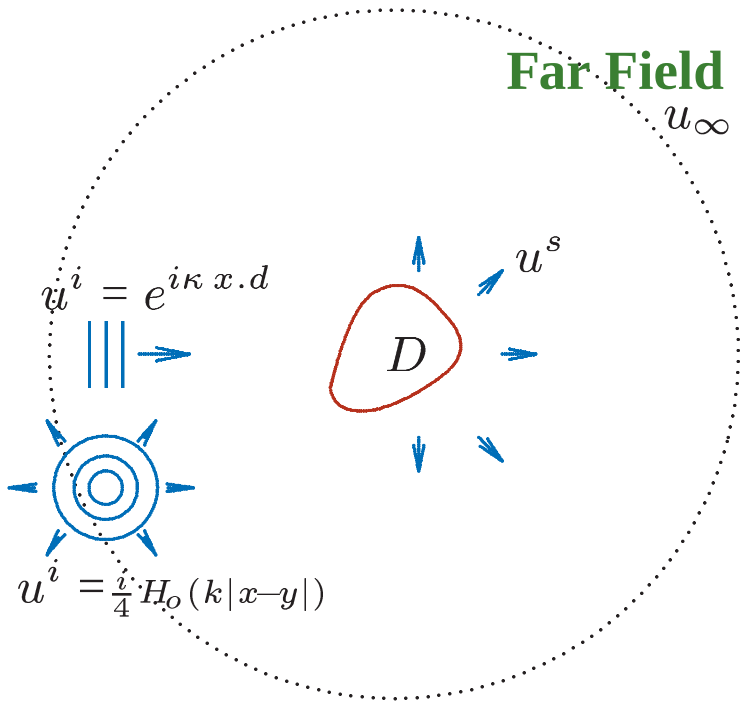

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$F : X \to L^2(S^1)$ maps a set X of admissible boundaries onto the far field pattern, $F(\partial D) = u_\infty$. F is nonlinear and *ill-posed*.

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Clearly, some compromises will have to be made

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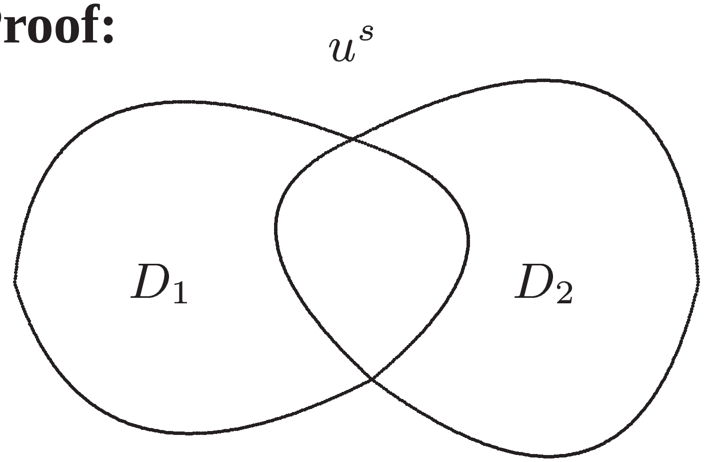

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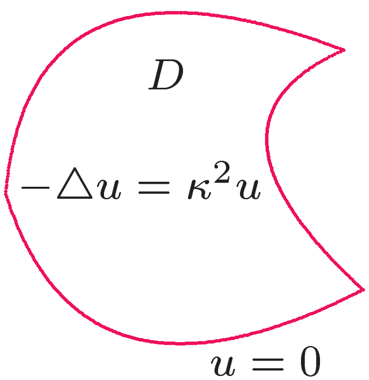

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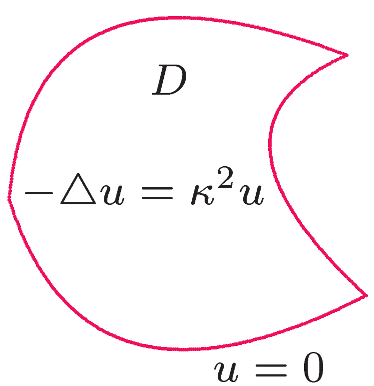

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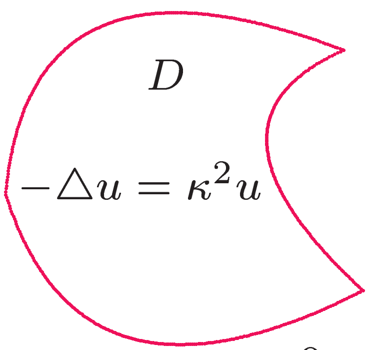

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(A similar result holds if there is an infinite number of plane waves with constant direction d, but κ varying over an finite interval.)

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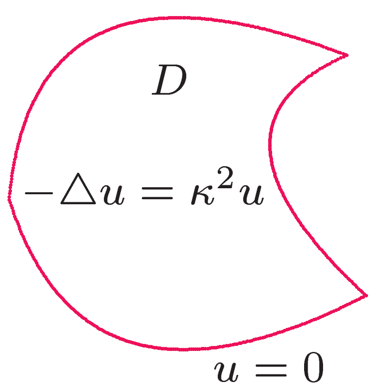

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What about other boundary conditions?

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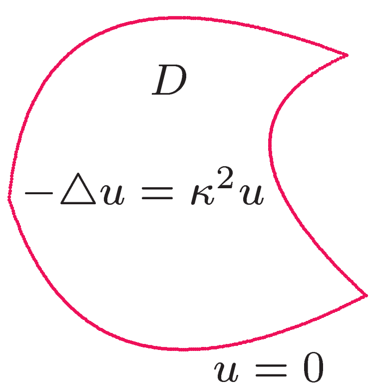

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Unfortunately, Schiffer's proof fails for non sound soft conditions due to the potential cusps in the domain D above. However . . . .

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But the information count is wrong! We are giving an *infinite* number of complex valued functions on S just to obtain *one real* curve ∂D.

It has been a long-standing conjecture that the far field pattern from a *single* incident plane wave suffices to determine an obstacle D.

Theorem: (Colton, Sleeman) *If* D_1 *and* D_2 *are two sound soft scatterers contained in a ball of radius* $< \pi/\kappa$ *and if the far field patterns coincide for a single incident plane wave with wavenumber* κ*, then* $D_1 = D_2$.

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Warning! Let the incident wave be a superposition of plane waves:

$$u^i(x) = \frac{\sin \kappa|x|}{|x|} = \frac{\kappa}{4\pi}\int_S e^{i\kappa\, x.d} ds(d)$$

Let D be a ball of radius R, center the origin. Then $u^s(x) = \frac{\sin \kappa R}{e^{i\kappa R}} \frac{e^{i\kappa|x|}}{|x|}$

Thus the total field $u(x) = \sin \kappa(|x| - R)e^{-i\kappa R}/|x|$ vanishes on the sphere with radius $R + m\pi/\kappa$ for all integers m.

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We must use special features of the incident wave.

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In this lecture we will concentrate on local uniqueness results; that is, we will attempt to prove injectivity of the derivative map F'. Since we will be using iteration schemes, such as Newton's method, this will be a necessary step.

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Typically such methods can give information under minimal amounts of data, but they have drawbacks; an initial ansatz (such as a single obstacle) and "reasonable" initial approximation to the solution is required.

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We mention one such approach: the *sampling method* due originally to Colton and Kirsch - with a critical mathematical refinement due to Kirsch.

Define the *far field operator* $G : L^2(S) \to L^2(S)$ by

$$G[f](x) = \int_S u_\infty(x,d) f(d) ds(d) \qquad x \in S.$$

This is an integral operator with kernel $u_\infty(x,d)$, which is the far-field pattern at a point x from an incident plane wave from angle d.
Further, let $\Psi_\infty(x,z)$ be the far field pattern of the fundamental solution:

$$\Psi_\infty(x,z) = \frac{1}{4\pi} e^{i\kappa x.z} \qquad x \in S, \quad z \in I\!R^{2,3}$$

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Colton and Kirsch gave mathematical reasons why one should expect that the solution of the above equation should have significantly larger norm for $z \notin D$ than for $z \in D$.

Led to a practical reconstruction method (and an industry!).

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Theorem: (Kirsch) *Assume that* κ *is not an eigenvalue for* $-\triangle$ *on the domain* D *subject to an impedance or transmission condition. Then* $z \in D$ *if and only if the ill-posed equation*

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Advantages:

- Requires (virtually) no apriori knowledge of the domain D.
- Is easy and fast to implement.

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Disadvantages:

- Requires enormous amounts of data; full far field pattern from all incident directions.
- Must solve a highly ill-conditioned integral equation whose kernel contains the data - very susceptible to noise.

Local Uniqueness Results

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For a reference domain D and a function $h \in C^2(\partial D) \to I\!R^n$, we define $\partial D_h := \{x + h(x) : x \in \partial D\}$, $B_r := \{h \in C^2(\partial D) : \|h\|_{C^2(\partial D)} < r\}$ then the *boundary to far field map* F, is defined by $F : B_r \to L^2(S)$.

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Since v satisfies the same boundary value problem as the function u (with different Dirichlet values) it can be computed very quickly from u. The largest part of the computation of F and F' is the common inversion of the matrix representing $\triangle v + k^2 v$.

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Proof: Let $F'(\partial D) = 0$. Then $v_\infty = 0$ and by Rellich's lemma, $v = 0$ in $I\!R^n \backslash \bar{D}$ and so $v = 0$ on ∂D. Thus $h_\nu \frac{\partial u}{\partial \nu} = 0$ on ∂D. However, by Holmgren's theorem, $\frac{\partial u}{\partial \nu}$ cannot vanish on any arc of ∂D since this would mean u is identically zero in $I\!R^n \backslash \bar{D}$. Hence $h_\nu = 0$.

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The Jacobi–Anger expansion gives

$$u^i(x) = e^{i\kappa\, x\cdot d} = \sum_{n=-\infty}^{\infty} i^n\, J_n(\kappa\rho) e^{in(\theta-\theta_0)}, \quad x \in \mathbb{R}^n,$$

From this it can be seen that the total field $u = u^i + u^s$ has the form

$$u(x) = \sum_{n=-\infty}^{\infty} \frac{i^n}{H_n^{(1)}(\kappa)} \{J_n(\kappa\rho) H_n^{(1)}(\kappa) - J_n(\kappa) H_n^{(1)}(\kappa\rho)\} e^{in(\theta-\theta_0)}$$

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From the asymptotics of the Hankel functions we obtain

$$F(\theta) = u_\infty = -e^{-\frac{\pi i}{4}} \sqrt{\frac{2}{\pi\kappa}} \sum_{n=-\infty}^{\infty} \frac{J_n(\kappa)}{H_n^{(1)}(\kappa)}\, e^{in(\theta-\theta_0)}, \quad 0 \le \theta \le 2\pi.$$

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We compute $\frac{\partial u}{\partial \nu}\,(x)$ and then v in the above theorem to obtain

$$F'[1]\, h = \sqrt{\frac{8}{\pi\kappa^3}} \sum_{n=-\infty}^{\infty} \Big(\sum_{m=-\infty}^{\infty} h_m \, \frac{i^{-m}\, e^{i(m-n)\theta_0}}{H_{n-m}^{(1)}(\kappa)} \Big) \frac{e^{in\theta}}{H_n^{(1)}(\kappa)}$$

in terms of the Fourier coefficients $h_0, h_{\pm 1}, \ldots,$ of the perturbation h_ν.

For fixed N, the quasi-Newton scheme to recover the leading Fourier coefficients $h_0, h_{\pm 1}, \ldots, h_{\pm N}$ of the perturbation h_ν from the corresponding Fourier coefficients of the far field pattern u_∞^n has the form

$$\mathbf{A}_{nm} h_m = \mathbf{B}_n (u_n - u_\infty^n)$$

where **A** is a (complex) $2N+1 \times 2N+1$ matrix, **B** a $2N+1$ vector

$$\mathbf{A}_{nm}(\kappa) = \sum_{m=-N}^{N} \frac{e^{im(\theta_0 - \frac{\pi}{2})}}{H^{(1)}_{n-m}(\kappa)} \qquad \mathbf{B}_n(\kappa) = \sqrt{\frac{\pi^3 \kappa}{8}}\, e^{i(n\theta_0 - \frac{\pi}{4})} H^{(1)}_n(\kappa)$$

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Doesn't this mean that the inversion is well-posed? The bad news,

Right hand side $|B_N(\kappa)|$

$N \setminus \kappa$	0.5	1.0	1.5	2.0	2.5	3.0
5	7.9×10^{3}	2.6×10^{2}	3.7×10^{1}	9.9	3.8	1.9
10	1.2×10^{11}	1.2×10^{8}	2.2×10^{6}	1.3×10^{5}	1.5×10^{4}	2.6×10^{3}
15	3.0×10^{19}	9.3×10^{14}	2.2×10^{12}	3.0×10^{10}	1.1×10^{9}	7.4×10^{7}

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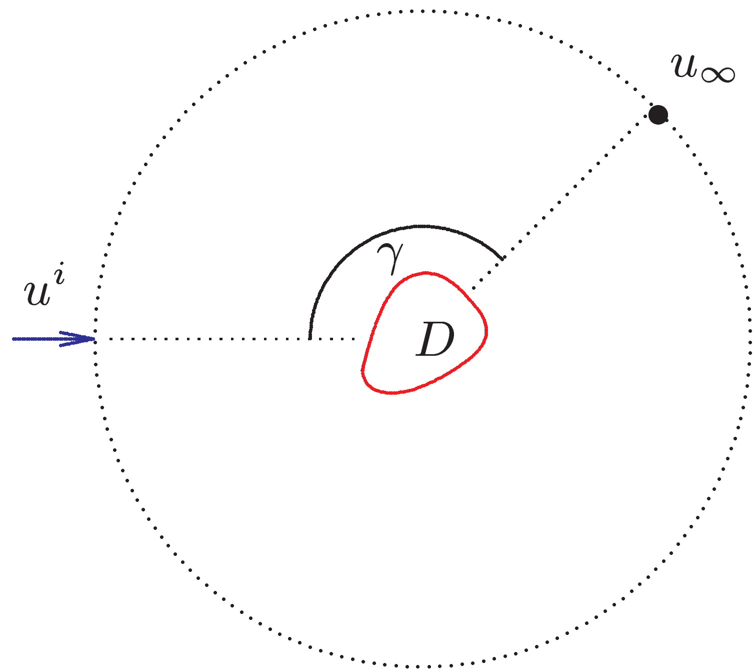
- Using additional frequencies κ does not allow more Fourier coefficients to be computed (should use largest κ).
- Our invertibility theorem shows the linear system is uniquely invertible at each step; no such guarantee can be given for full-Newton.
- We actually find that this frozen-Newton scheme gives reconstructions that are virtually identical to a full-Newton implementation.

Single Frequency, Multiple Direction Incident Waves

For a *fixed* offset angle γ and each direction d, $0 \leq d \leq 2\pi$, measure the far field pattern $u_\infty(d)$ at the single point given by the direction $d + \gamma$.
The frequency κ is fixed.

$\gamma = \pi$ is *backscattering* case,
$\gamma = 0$ is *forward scattering*.



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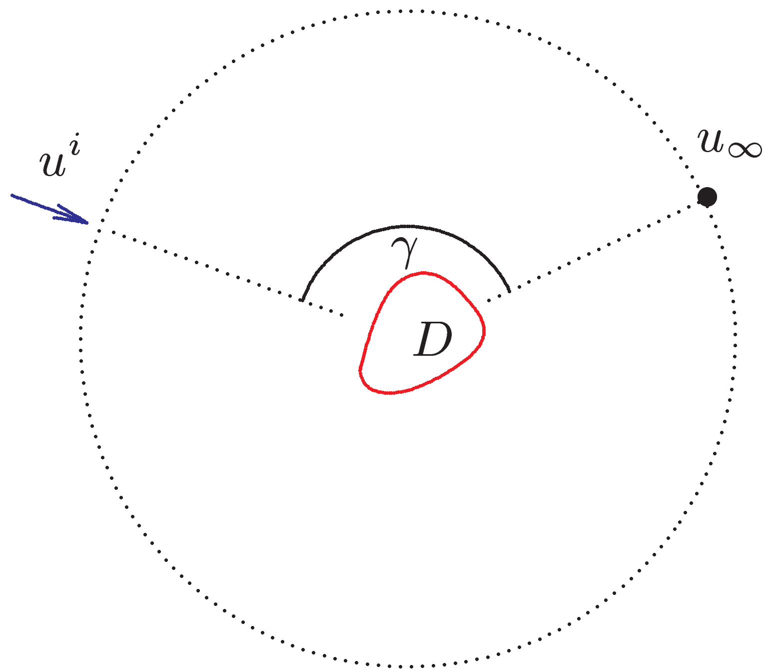

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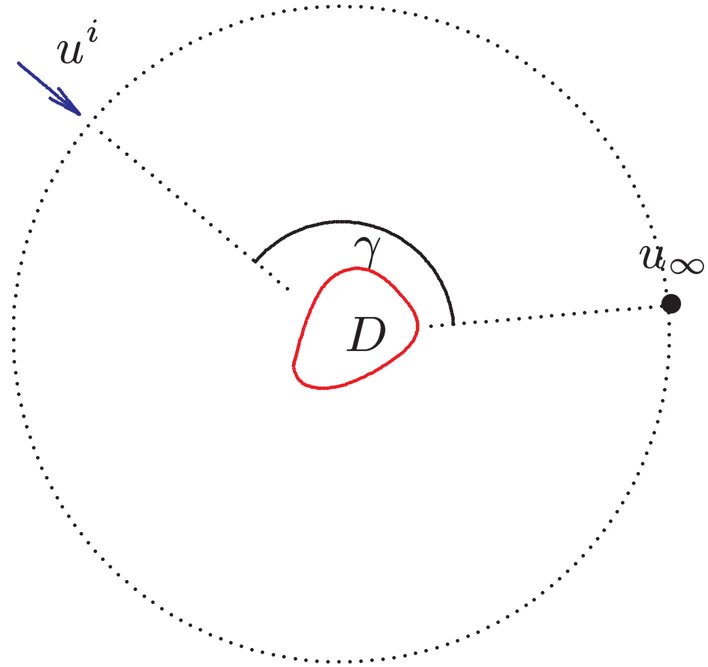

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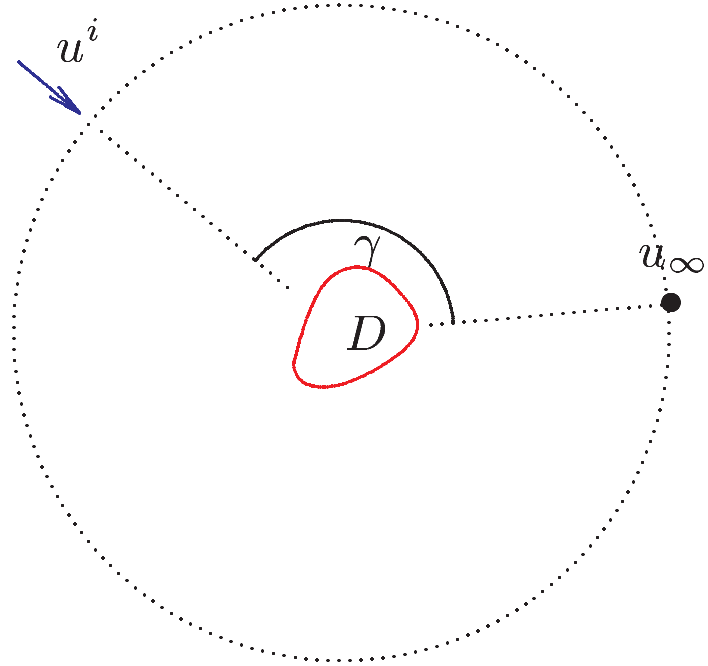

Theorem. Let $0 < \gamma < 2\pi$. Then if the wavenumber κ is sufficiently small the derivative map F' is injective. With M incident waves from distinct directions and a finite trigonometric basis the resulting Jacobian matrix has trivial nullspace provided $M > 2N + 1$.

In the forward scattering case, $\gamma = 0$, the odd cosine and sine coefficients of q (as measured from the origin) cannot be recovered.

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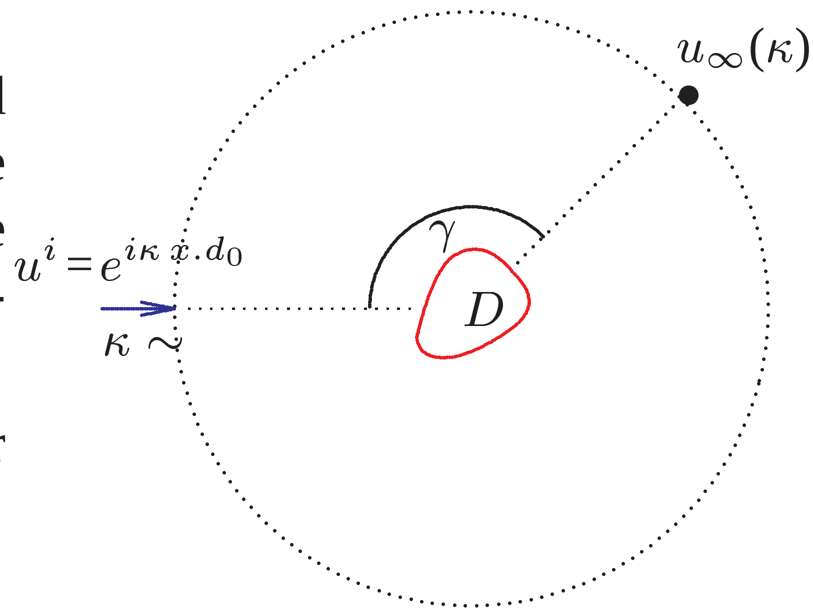
The frequency κ is varied over the range $[\kappa_{min}, \kappa_{max}]$.



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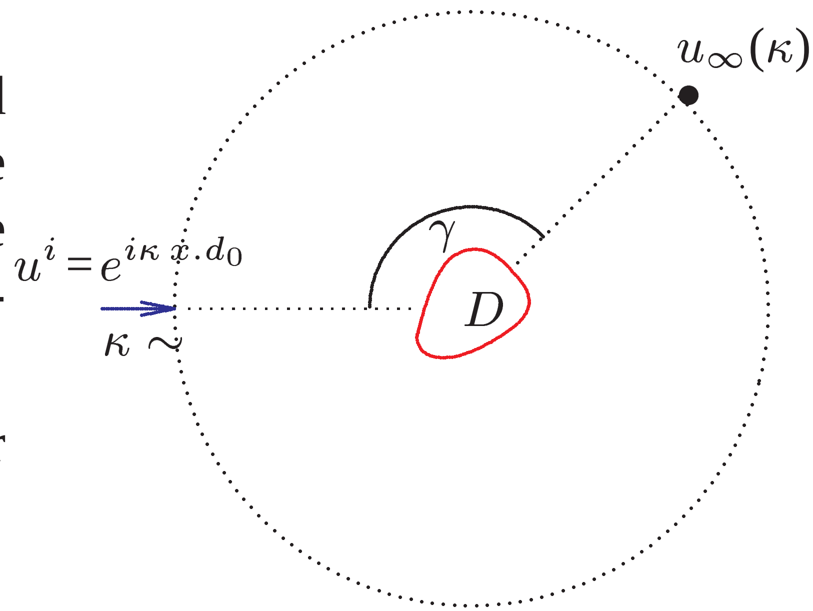

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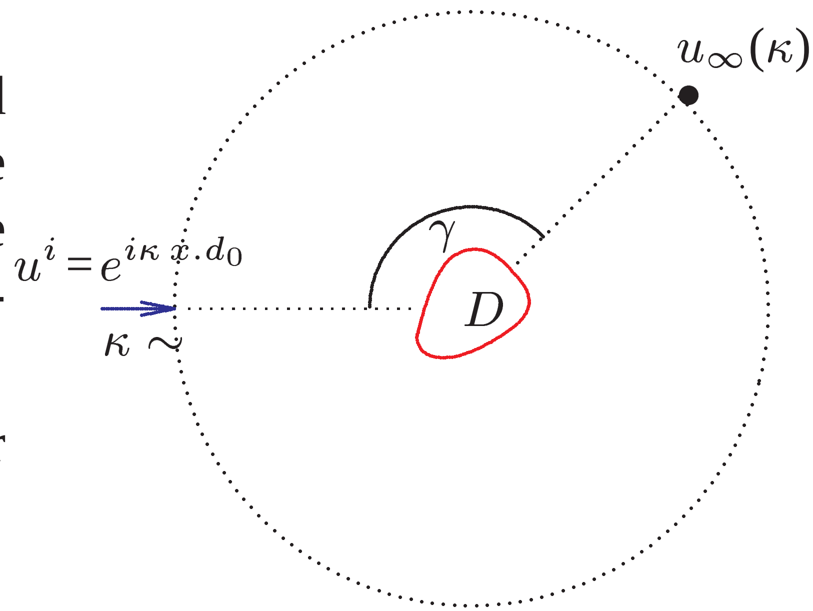

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Is there complete loss of information if $\gamma = 0$?

No, we can show that the nullspace of F' consists of all the sine coefficients in addition to all the odd cosine coefficients.

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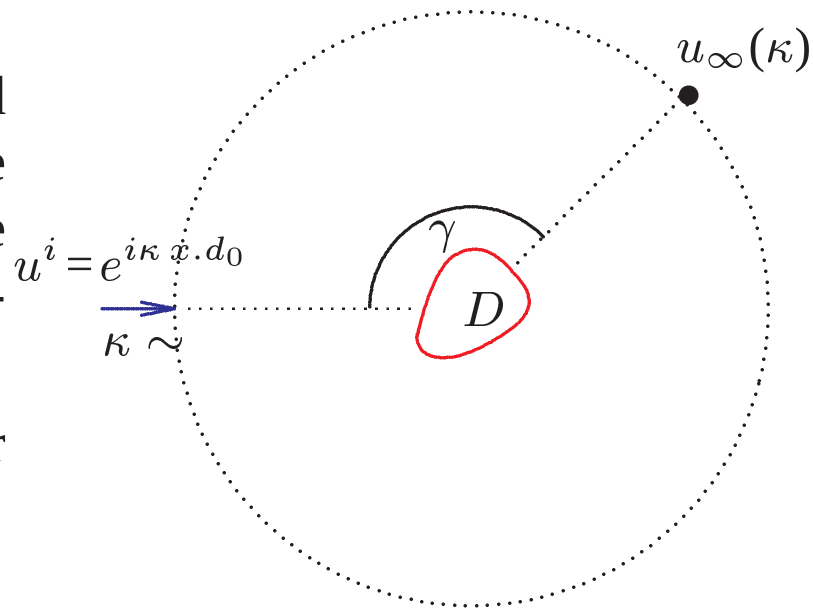

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The obvious question is, does measurements at a scan of frequencies at two distinct points recover full information?

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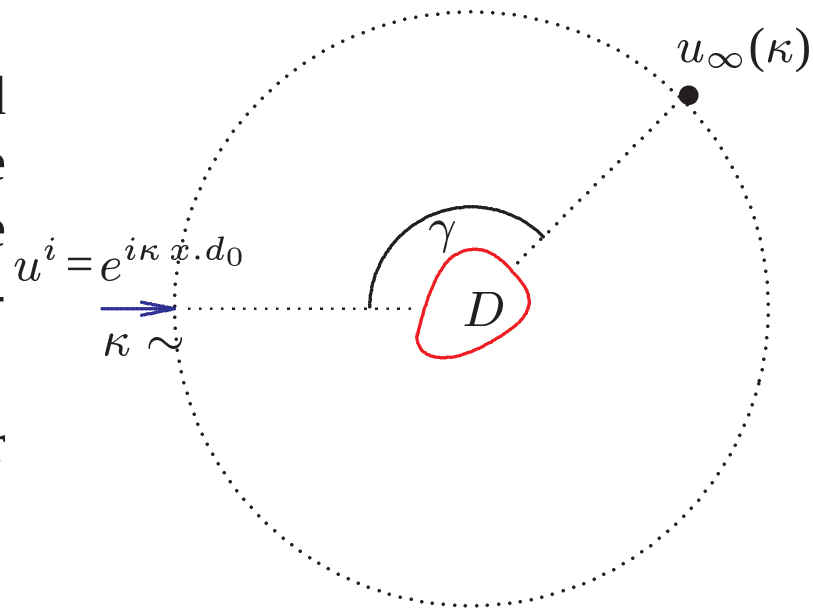

Theorem. For any γ with $0 < \gamma < 2\pi$, the dimension of the nullspace of F' is exactly N. The nullspace consists of the odd cosine and the even sine coefficients (with the origin at $t = \frac{\gamma}{2}$).

Corollary. From a measurement of the far field pattern at two angles γ_1 and γ_2 we can recover all Fourier coefficients of q provided $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$ and $\sin m\frac{(\gamma_1 - \gamma_2)}{2} \neq 0$, for $m = 1, \ldots, N$.

Impedance Boundary Conditions

$$\frac{\partial u}{\partial \nu} + i\lambda k u = 0 \qquad \text{on } \partial D.$$

On physical grounds $\lambda(\theta)$ is real and non-negative.

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What about local uniqueness for a single incident plane wave?

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Theorem. The map $F(\partial D, \lambda) \to u_\infty$ is Fréchet differentiable with respect to the boundary and the derivative $F_r' q$ in the direction h is given by the far field pattern $v_{r,\infty}$ of the solution v_r of the impedance boundary value problem with boundary condition

$$\frac{\partial v_r}{\partial \nu} + i\lambda k v_r = k^2\, h_\nu\, u + \frac{d}{ds}\left(h_\nu \frac{du}{ds}\right) - i\lambda k\, h_\nu \left(\frac{\partial u}{\partial \nu} - H u\right) \qquad (*)$$

H is the mean curvature on ∂D, $\quad h_\nu = \dfrac{\mathrm{h}\,\mathrm{q}}{\sqrt{\mathrm{q}^2 + (\mathrm{q}')^2}}$.

Theorem. Assume that $\lambda(x) > 0$ and $H(x) > 0$ for all $x \in \partial D$. Then the derivative of the map F with respect to the boundary is injective, i.e., $F_q' h = 0$ implies $h_\nu = 0$.

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Proof: If $F_q' h = 0$ then the solution to the scattering problem has vanishing far field pattern. Rellich's lemma and (*) shows

$$k^2(1-\lambda^2)h_\nu u + \frac{d}{ds}\left(h_\nu \frac{du}{ds}\right) + ik\lambda H h_\nu u = 0 \quad \text{on } \partial D,$$

multiply this by $\bar{u}$ and take the imaginary part to obtain

$$\frac{d}{ds}(h\nu U) = \lambda k H h\nu |u|^2 \qquad \text{on } \partial D,$$

where $U := \frac{i}{2}\, u\, \frac{d\bar{u}}{ds} - \frac{i}{2}\, \bar{u}\, \frac{du}{ds}$. Assume that h_ν is not identically zero. Then, the set $\Sigma := \{x \in \partial D : h_\nu(x) > 0\}$ is nonempty and

$$\int_\Sigma \lambda H h_\nu |u|^2 ds = 0.$$

Thus $u = 0$ on Σ. Holmgren's theorem gives a contradiction.

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The farfield u_∞ is a complex-valued function and we are only obtaining a real valued curve ∂D in return.

Theorem. Assume that $\lambda(x) > 0$ and $H(x) > 0$ for all $x \in \partial D$. Then the derivative of the map F with respect to the boundary is injective, i.e., $F'_q h = 0$ implies $h_\nu = 0$.

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Thus $u = 0$ on Σ. Holmgren's theorem gives a contradiction.

Is there any chance of recovering **both** ∂D and λ from a single measurement of u_∞?

Theorem. Assume that $\lambda(x) > 1$ for all $x \in \partial D$. Then the total derivative of the map F is injective, i.e., $F_q' h + F_\lambda' \mu = 0$ implies $h_\nu = 0$ and $\mu = 0$.

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Proof: Assume that $F'_r q + F'_\lambda \mu = 0$. Then again using by Rellich's lemma we have $v = 0$ in $I\!R^n \setminus \bar{D}$ and therefore

$$k^2(1-\lambda^2)h_\nu u + \frac{d}{ds}\left(h_\nu \, \frac{du}{ds}\right) + ik(\lambda H h_\nu - \mu)u = 0 \quad \text{on } \partial D,$$

Multiplying this by $\bar{u}$ and taking the real part we obtain

$$k^2(1-\lambda^2)h_\nu|u|^2 - h_\nu\left|\frac{du}{ds}\right|^2 + \frac{1}{2}\frac{d}{ds}\left(h_\nu \, \frac{d|u|^2}{ds}\right) = 0 \quad \text{on } \partial D.$$

Assume that h_ν is not identically zero. Then, without loss of generality, the set $\Sigma := \{x \in \partial D : h_\nu(x) > 0\}$ is nonempty and therefore

$$\int_\Sigma \left\{ k^2(1-\lambda^2)|u|^2 - \left|\frac{du}{ds}\right|^2 \right\} h_\nu \, ds = 0.$$

Since $\lambda(x) > 1$ for all $x \in \partial D$ we obtain $u = 0$ on Σ. By the boundary condition for u this implies $\partial u / \partial \nu = 0$ on Σ and employing Holmgren's theorem we arrive at the contradiction that $u = 0$ in $I\!R^n \setminus D$.

Theorem. Assume that $\lambda(x) > 1$ for all $x \in \partial D$. Then the total derivative of the map F is injective, i.e., $F'_q h + F'_\lambda \mu = 0$ implies $h_\nu = 0$ and $\mu = 0$.

Is the assumption $\lambda > 1$ necessary?

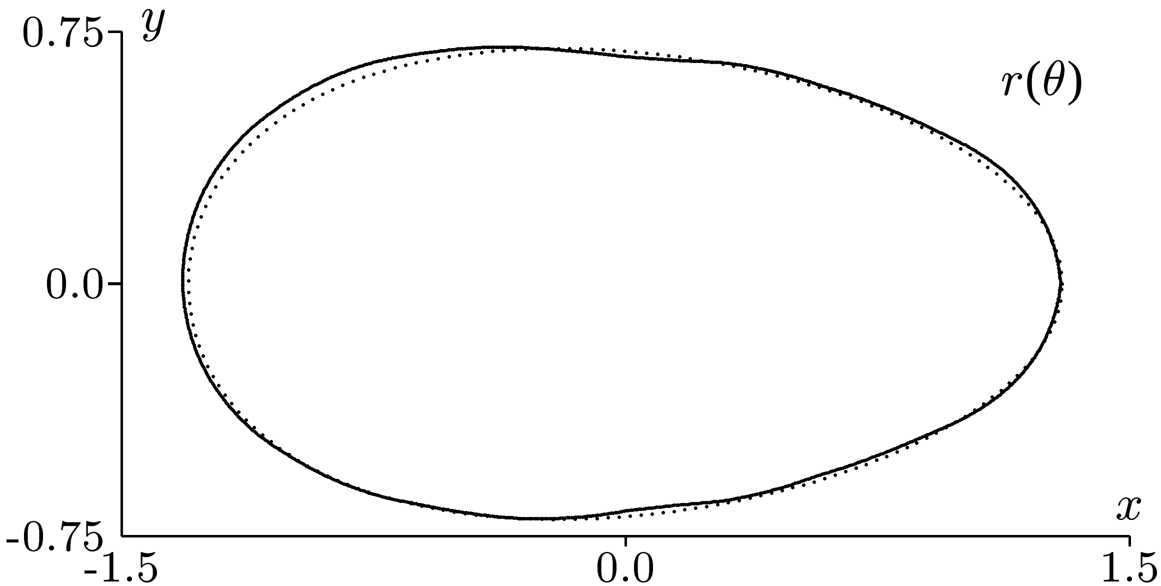
Theorem. Assume that $\lambda(x) > 1$ for all $x \in \partial D$. Then the total derivative of the map F is injective, i.e., $F'_q h + F'_\lambda \mu = 0$ implies $h_\nu = 0$ and $\mu = 0$.

Is the assumption $\lambda > 1$ necessary?

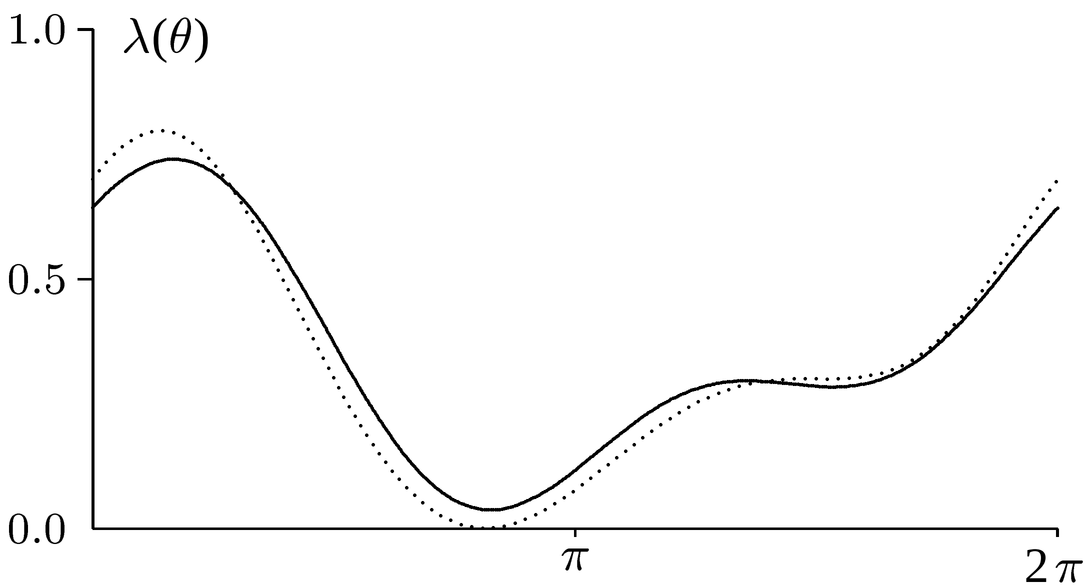
For a plane wave incident field we suspect not. However, for the cylindrical wave $u^i(x) = J_n(k|x|)e^{in\, arg\, x}$ we can show that the resulting total wave satsifies the radiating condition, the Helmholtz equation and if $0 \leq \lambda < 1$, for $h_\nu = 1$ the critical equation is also satisfied

$$k^2(1-\lambda^2)h_\nu|u|^2 - h_\nu\left|\frac{du}{ds}\right|^2 + \frac{1}{2}\frac{d}{ds}\left(h_\nu\,\frac{d|u|^2}{ds}\right) = 0 \quad \text{on } \partial D.$$

Thus studying this equation without regard to the incident wave is **not** enough.



0.75
y
$r(\theta)$
0.0
-0.75
x
-1.5
0.0
1.5

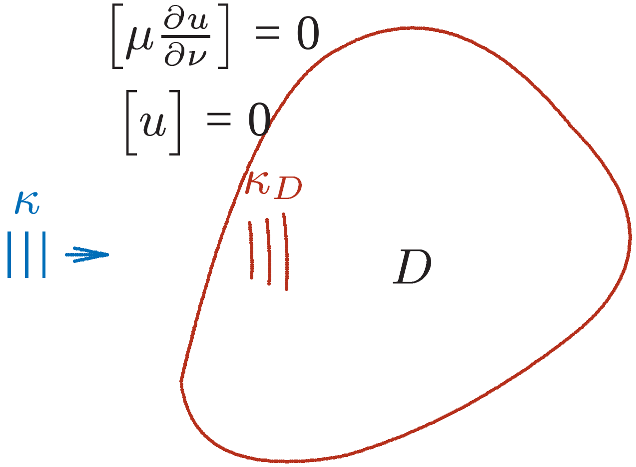


1.0
$\lambda(\theta)$
0.5
0.0
π
2π

Transmission boundary conditions

$$\left[\mu \frac{\partial u}{\partial \nu}\right] = 0$$

$$\left[u\right] = 0$$



On ∂D the total field satisfies the conditions

$$u\big|_+ = u\big|_-$$

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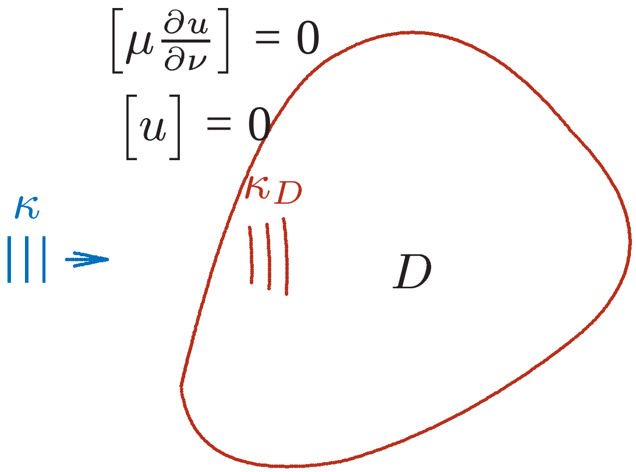

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The map $F : \partial D \to u_\infty$ is Fréchet differentiable and $F'.h$ is given by the far field pattern u'_∞ of the radiating solution to

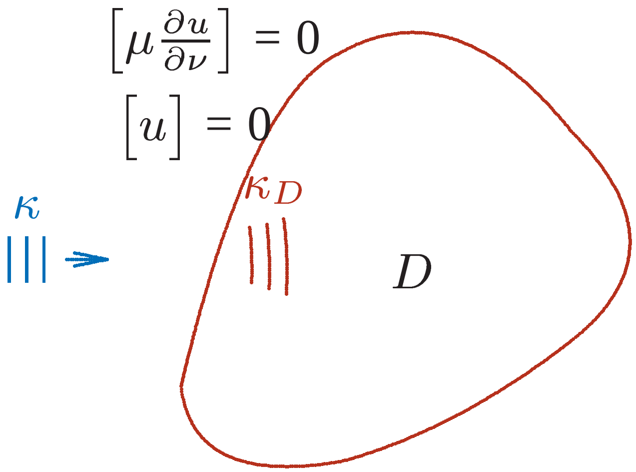
$$-\triangle u' + k_D^2 u' = 0 \qquad \text{in } D \qquad -\triangle u' + k^2 u' = 0 \qquad \text{in } R^n/D$$

satisfying the Sommerfeld condition and the boundary conditions

$$[u']_\pm = \frac{1-\mu}{\mu}\frac{\partial u}{\partial \nu}h_\nu \qquad \left[\mu\frac{\partial u'}{\partial \nu}\right]_\pm = (k^2 - \mu k_D^2)uh_\nu + (1-\mu)\mathrm{Div}(h_\nu \nabla_\tau u)$$

$\mathrm{Div}\, u$ and $\nabla_\tau u$ are the surface divergence and gradients respectively.

Transmission boundary conditions



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Uniqueness questions here are largely open.

Iteration Schemes

We attempt to solve the nonlinear equation $F(x) = g$ by an iterative method of the form

$$x_{n+1} = x_n + \mathcal{A}_n(F(x_n) - g)$$

Some possible choices for $\mathcal{A}$ are:

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Let's look at one possibility for the latter case.

Suppose we have the map $F(x) = g$ and a starting approximation x_0. To compute the step $n+1$, let $\tilde{h}$ be computed by a Newton step,

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A **linear** equation. The next iteration $x_{n+1} = x_n + h$ is defined from the 2nd degree Taylor remainder by the solution of the **linear** equation

$$F'[x_n]h + \tfrac{1}{2}F''[x_n](\tilde{h}, h) = g - F(x_n).$$

Iteration Schemes

We thus obtain a *predictor-corrector* scheme:

$$\tilde{h} = (F'[x_n])^{-1}(g - F(x_n)) \qquad \text{Predictor}$$

$$h = \left(F'[x_n] + \tfrac{1}{2}F''[x_n](\tilde{h}, \cdot)\right)^{-1}(g - F(x_n)) \qquad \text{Corrector}$$

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Theorem. *Let* $\hat{x} \in U \subseteq X$ *denote a solution of (1). Assume* $F'[\hat{x}]$ *admits a bounded inverse and* F' *and* F'' *are uniformly bounded in* U. *Then there exists* $\delta > 0$ *such that the above iteration with starting guess* $x_0 \in B(\hat{x},\delta) = \{x \in X : \|\hat{x} - x\| < \delta\}$ *converges quadraticly to* $\hat{x}$. *Additionally, if the second derivative is Lipschitz continuous, i.e.*

$$\|F''[x](h,\tilde{h}) - F''[y](h,\tilde{h})\| \leq L\|x - y\|\,\|h\|\,\|\tilde{h}\|$$

for all $x, y \in U$ *with* $h, \tilde{h} \in X$ *and a constant* $L > 0$, *then*

$$\|x_{n+1} - \hat{x}\| \leq c\,\|x_n - \hat{x}\|^3$$

holds for $n = 0, 1, 2, \ldots$ *with a constant* $c > 0$.

Back to Scattering

Theorem. *F is twice differentiable. F' is represented by the far field pattern $F'[\partial D]\, h = u'_\infty$ of the solution u' of the exterior Dirichlet problem,*

$$\triangle u' + k^2 u' = 0 \quad in\ \mathbb{R}^n \setminus \bar{D} \qquad u' = -h_\nu \frac{\partial u}{\partial \nu} \quad on\ \partial D$$

$F''[\partial D](h_1, h_2) = u''_\infty$ *where* u''_∞ *is the far field pattern of the radiating solution* $u'' \in H^1_{loc}(\mathbb{R}^n \setminus \bar{D})$ *of the exterior Dirichlet problem*

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$$+ \left(h_{1,\tau}(\tau \cdot \nabla(h_{2,\nu})) + h_{2,\tau}(\tau \cdot \nabla(h_{1,\nu}))\right) \frac{\partial u}{\partial \nu} \quad \text{on } \partial D$$

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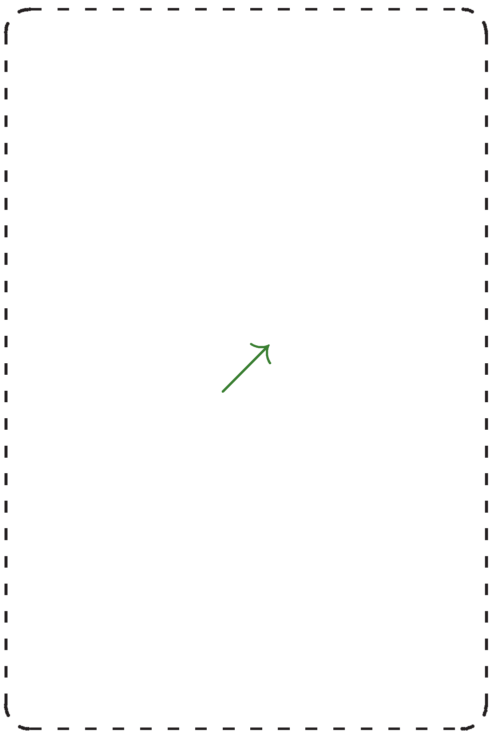
Note: The largest part of the computation of F, F' and F'' is the common inversion of the matrix representing $\triangle v + k^2 v$.

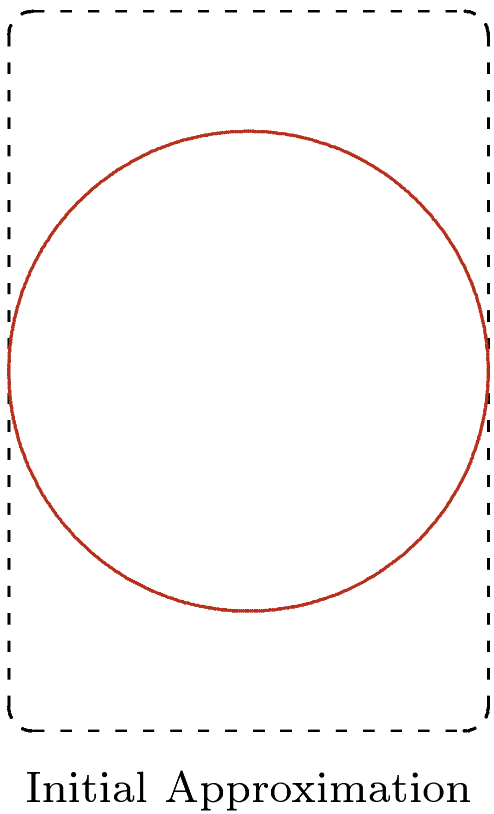


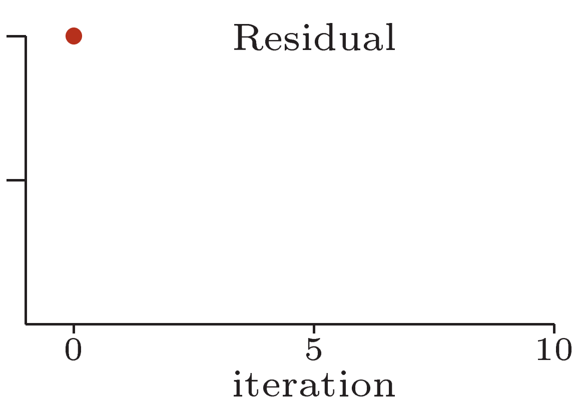
Exact Obstacle

Direction of Incident Wave

Reconstruction of a sound soft rectangular object from a single incident field using accurate data ($\sim 0.1\%$ noise).

The size of the obstacle is 1.2×1 units and the value of κ is one.

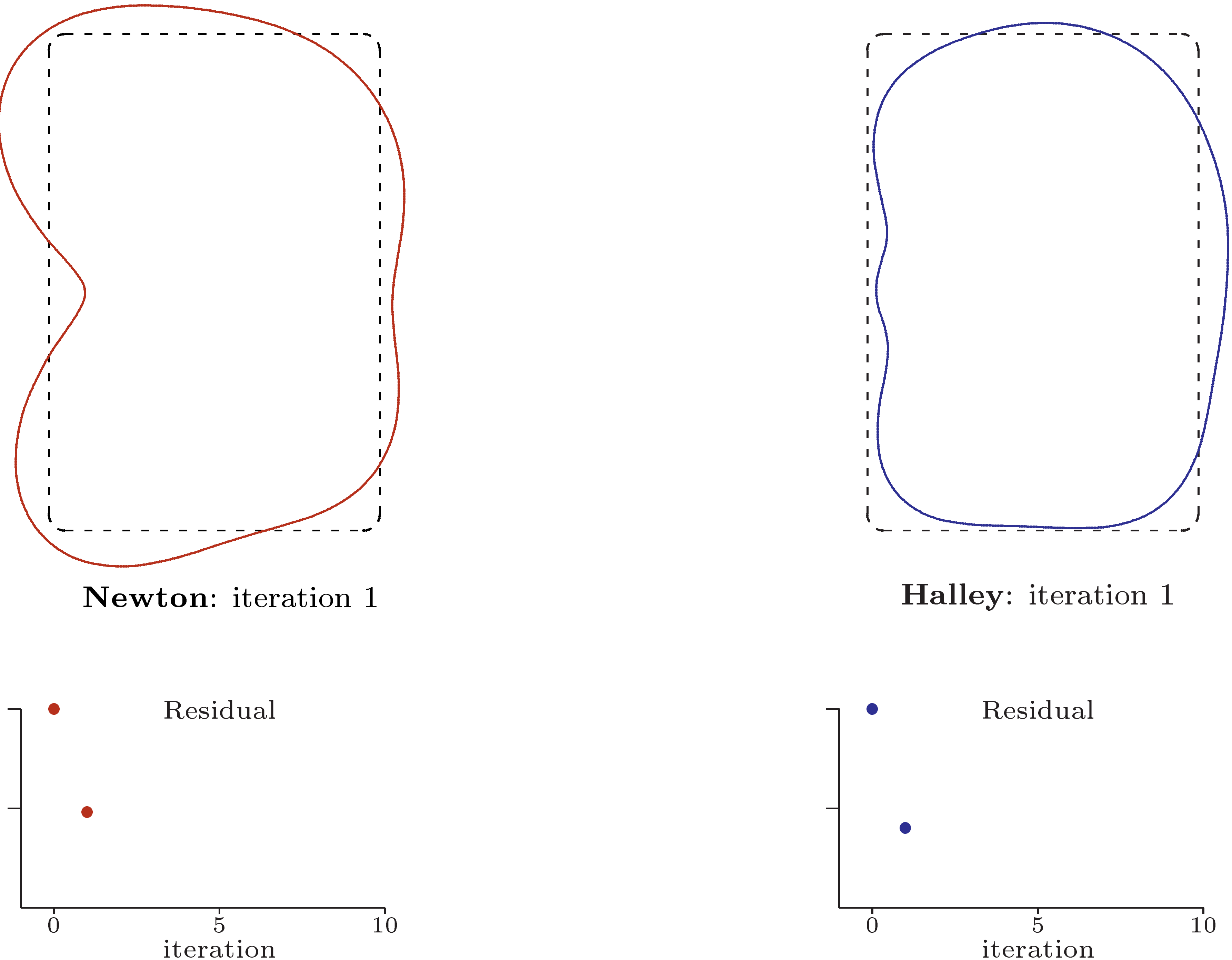




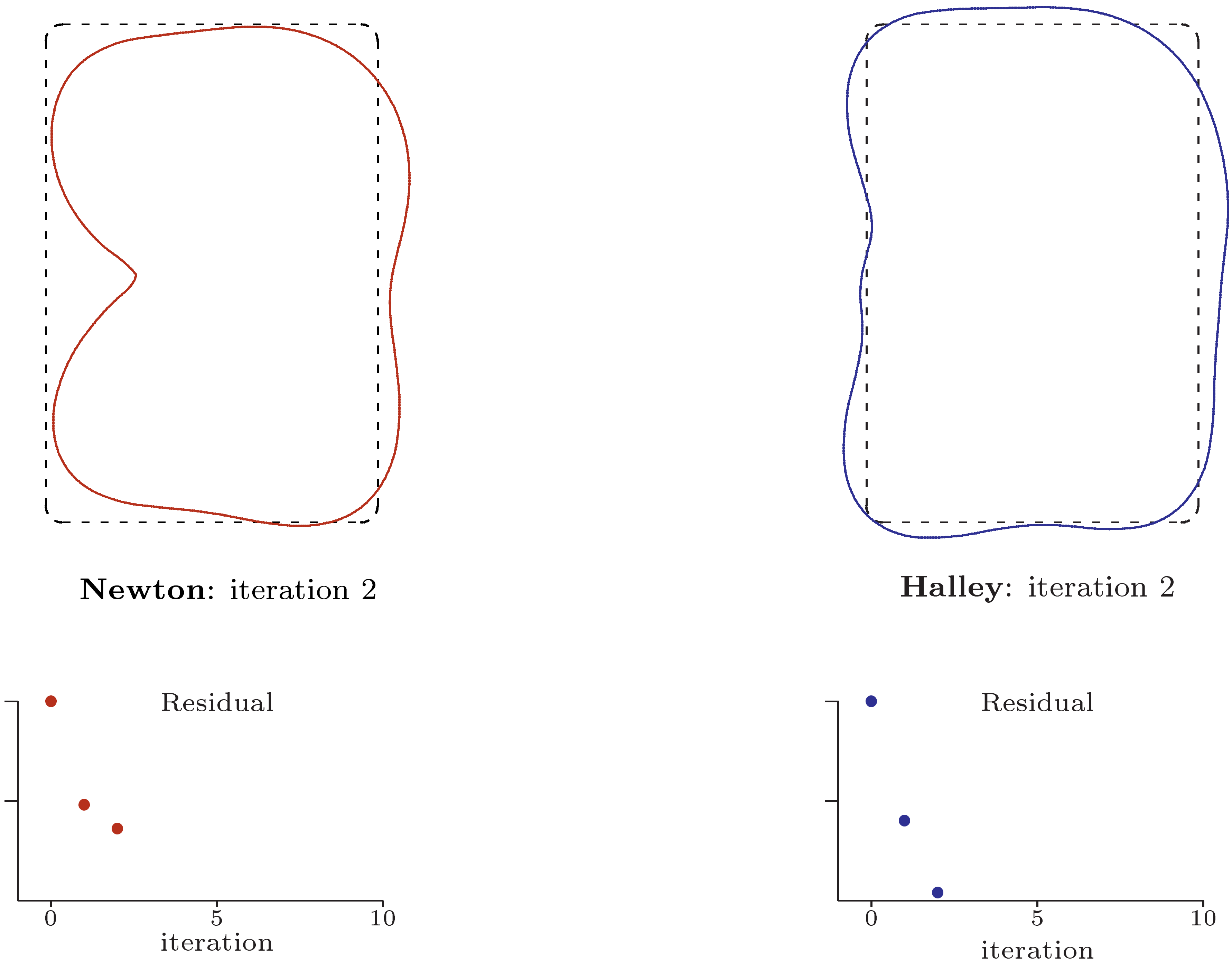
The graphic below plots the relative residual is against iteration number, $\|F(q_n) - u_\infty\| / \|F(q_0) - u_\infty\|$

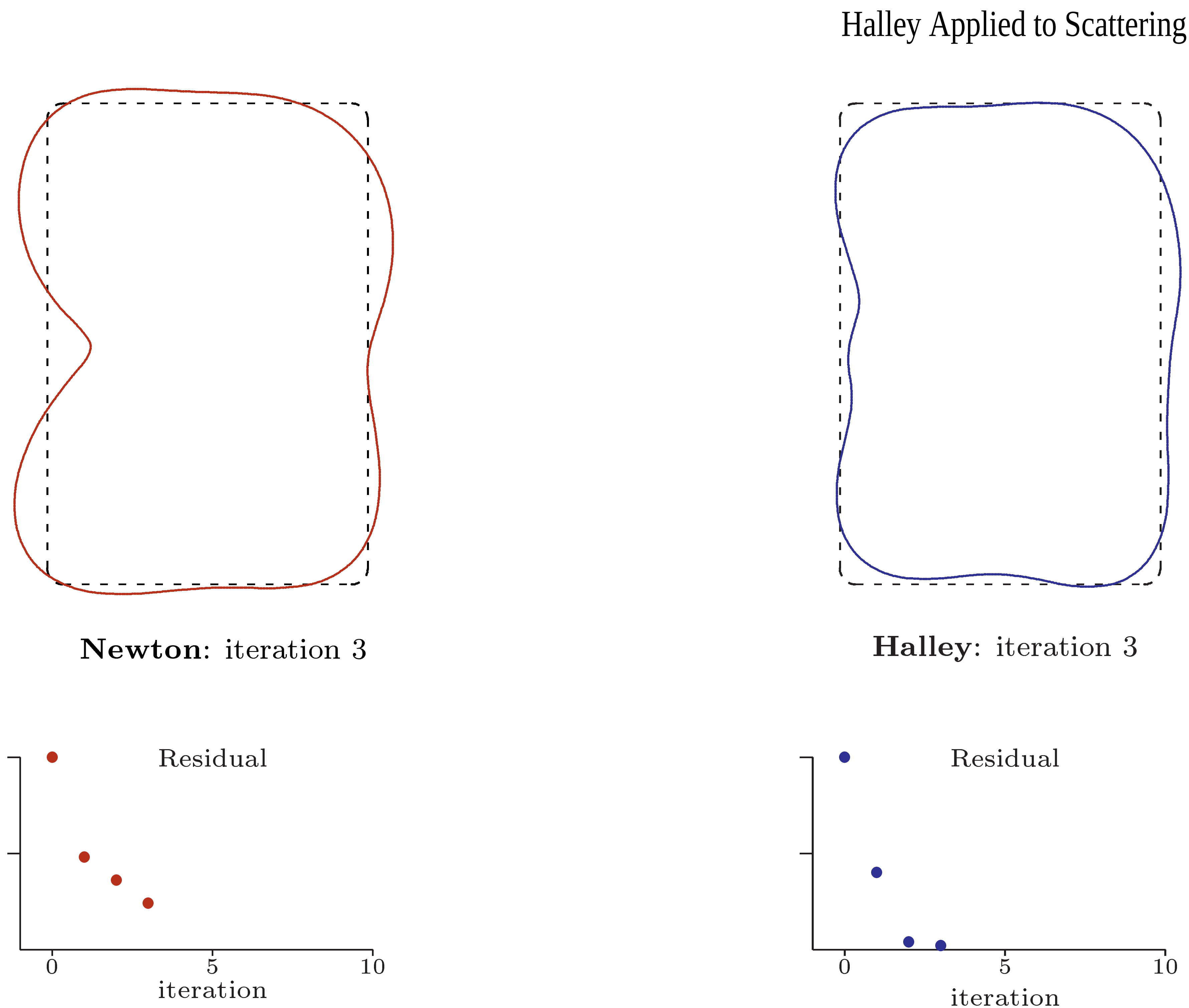
Tichonov regularisation is used.

Halley Applied to Scattering



Halley Applied to Scattering




Halley Applied to Scattering
Newton: iteration 3
Halley: iteration 3
Residual
0
5
10
iteration
Residual
0
5
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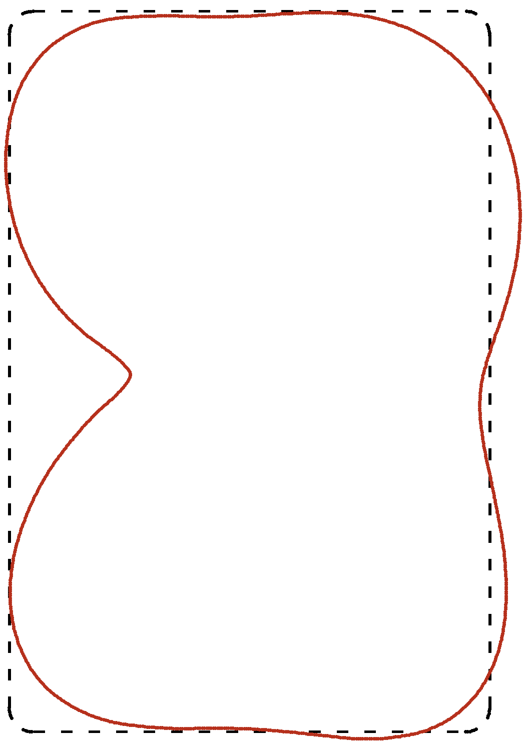
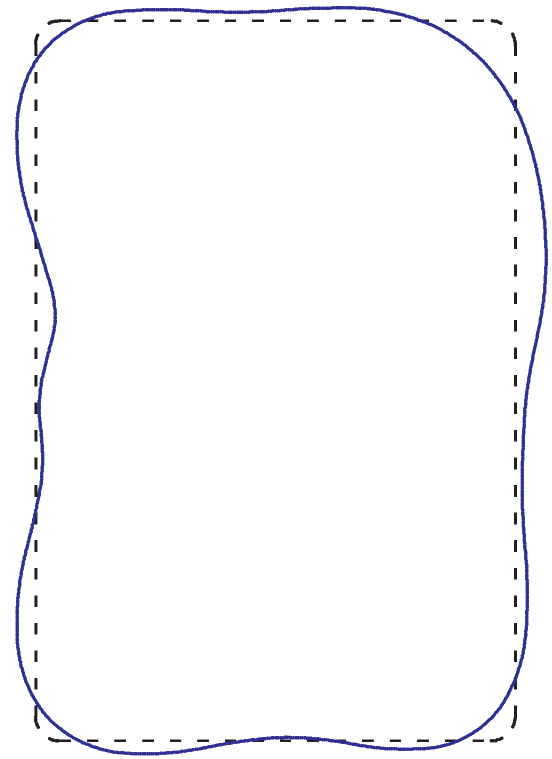
iteration

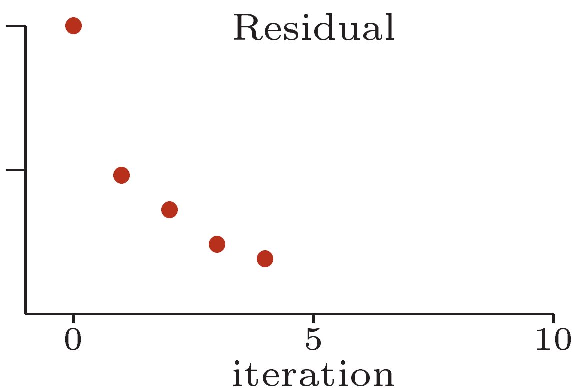
Halley Applied to Scattering

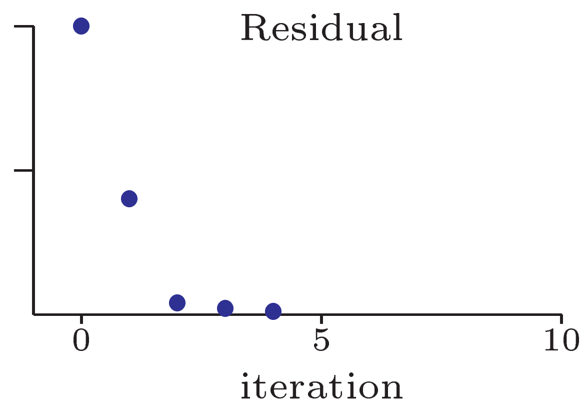


Newton: iteration 4

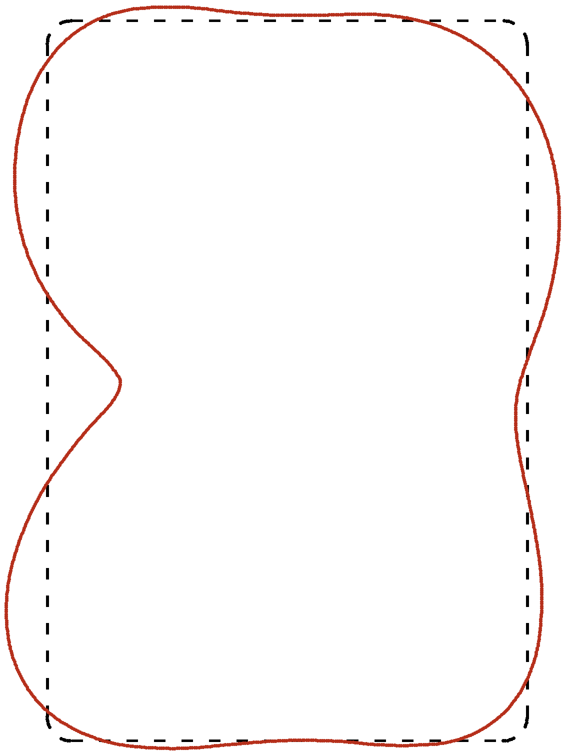


Halley: iteration 4

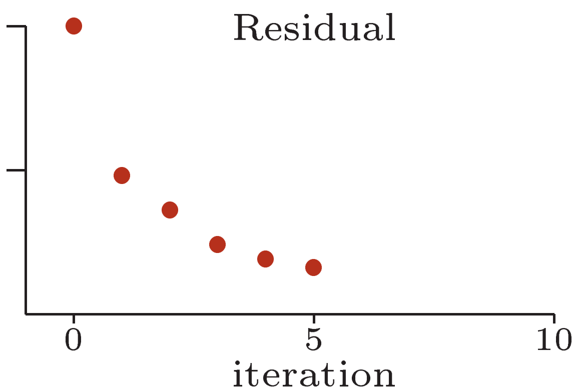


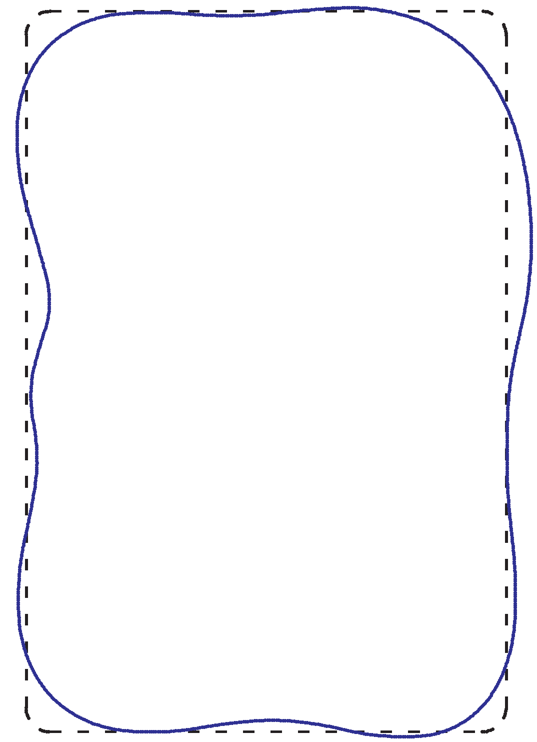


Halley Applied to Scattering

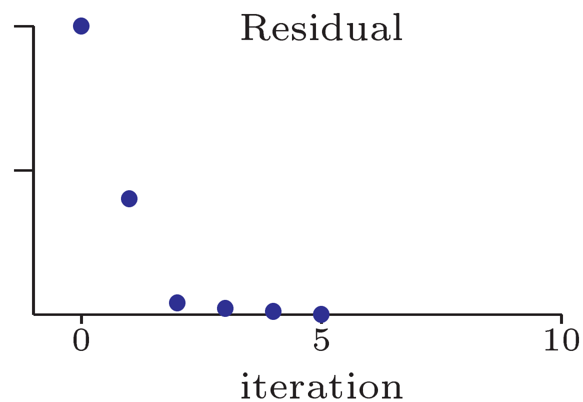


Newton: iteration 5

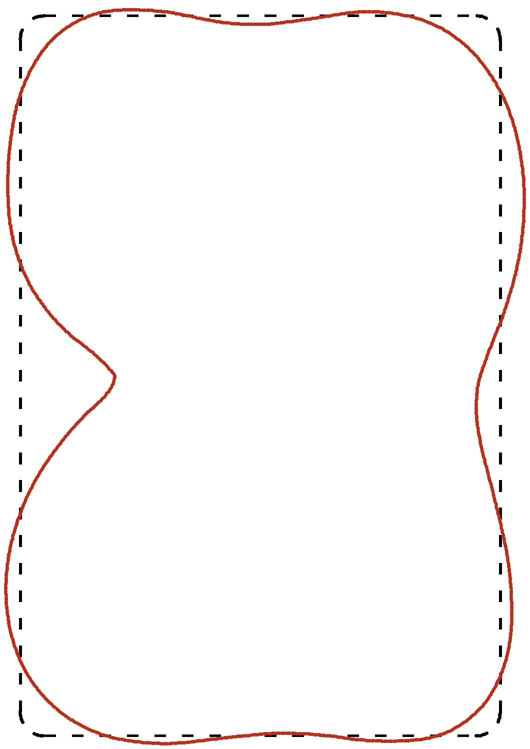




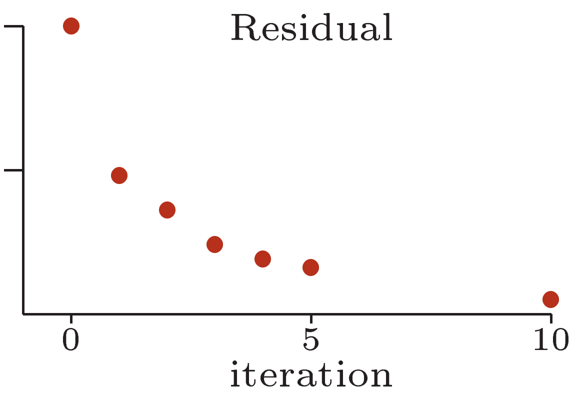
Halley: iteration 5

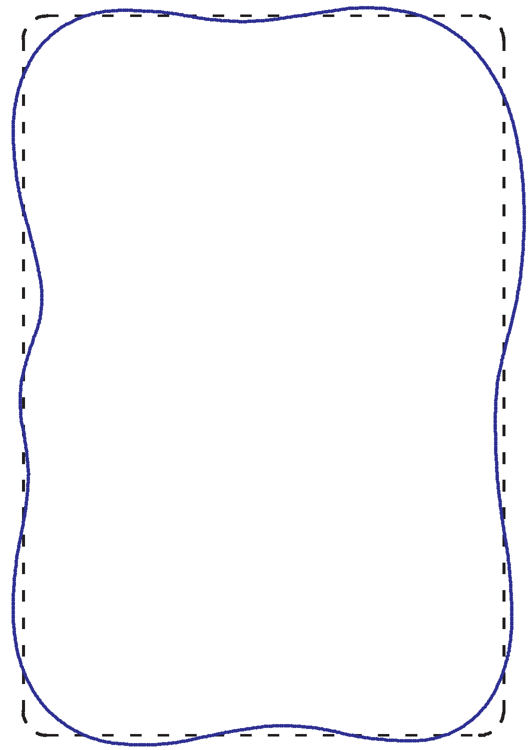


Halley Applied to Scattering

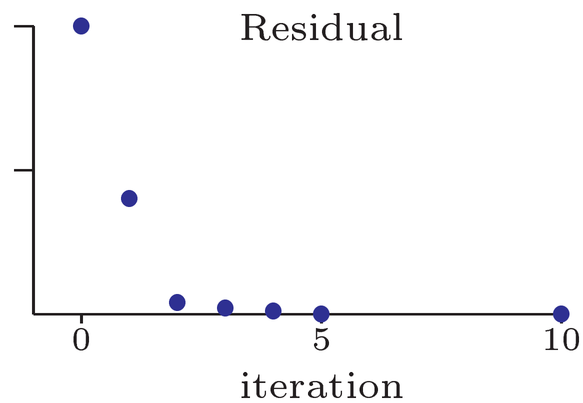


Newton: iteration 10

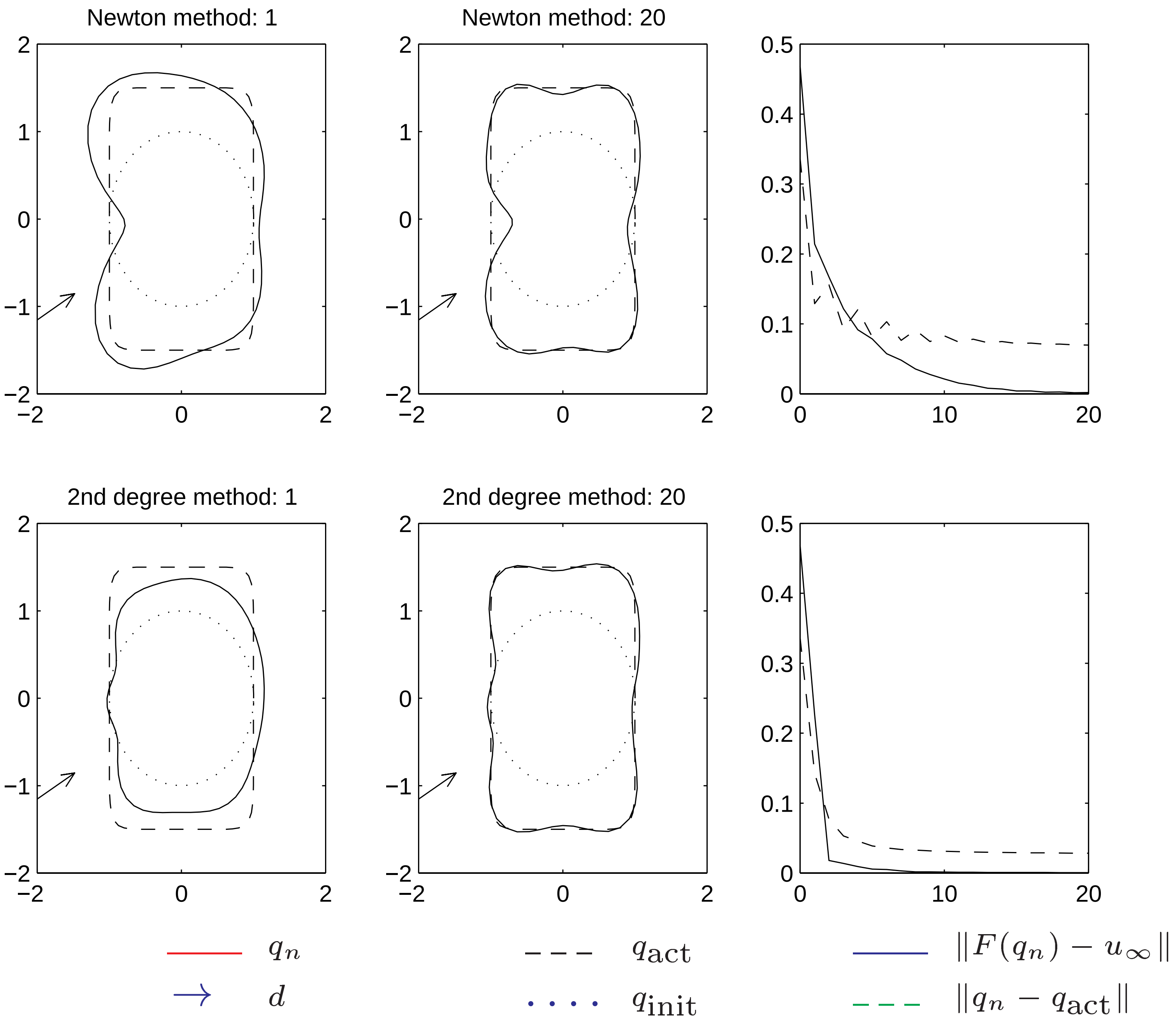




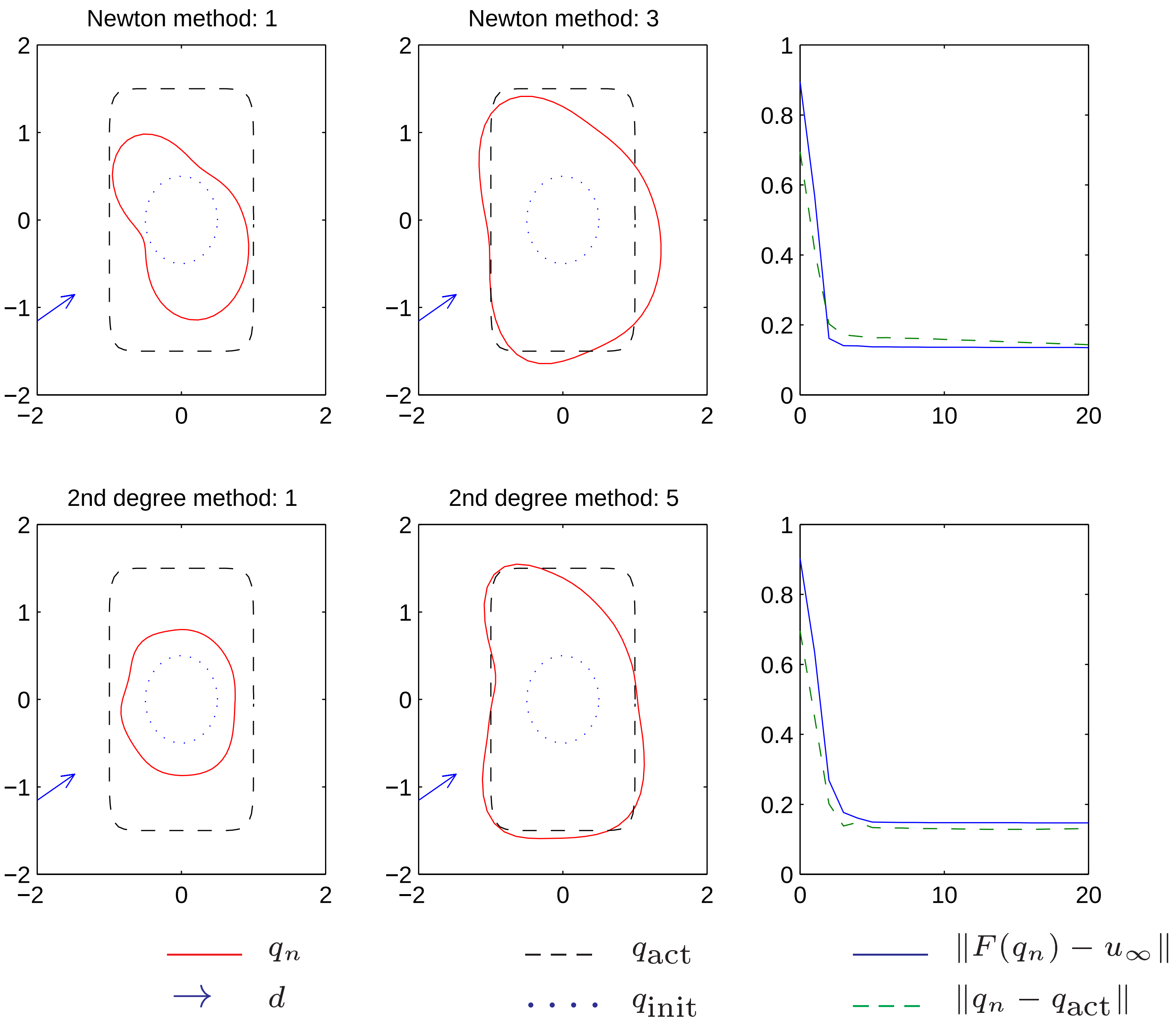
Halley: iteration 10



Reconstructions from noise-free data



Reconstructions from data with 10% noise



Some Open Issues

We are solving the Newton equation $F'[x]\,\tilde{h} = g - F(x)$ where $F'[x]$ is singular. Let us suppose that it turns out that F' is a positive semidefinite matrix (this may come from the maximum principle in the underlying pde) and is also symmetric (just to simplify the idea and avoid transposes). Then we might regularise by

$$(F'[x] + \alpha R)\,\tilde{h} = g - F(x)$$

where R is a positive (semi-definite) matrix and $\alpha > 0$.

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where R is a positive (semi-definite) matrix and $\alpha > 0$.

- What if (again from the underlying pde) it could be shown that the Hessian $F''[x](\,\cdot\,, h)$ was a positive semidefinite matrix? Would then the choice $R = \frac{1}{2}F''[x](\,\cdot\,, h)$ be a regularisation?

  Even if it only partly (we cannot expect full) did so, it might allow a reduction in the level of regularisation required by another scheme.

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  Even if it only partly (we cannot expect full) did so, it might allow a reduction in the level of regularisation required by another scheme.

- Can we effectively use a higher (than two) order MacLaurin expansion in an attempt to better model the nonlinear map? For most problems we think the answer is no. However, we were once convinced that a single derivative was the sensible limit.

Is Halley’s method a Panacea?

We have painted a very rosy picture of the power of Halley’s method. To be fair we should point out difficulties.

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- Regularisation issues are more complex. With any regularisation method there is always the tricky problem of choosing the optimal value of the regularisation parameter With the second degree method we have to select two regularisation parameters, one each for the predictor and the corrector; experience shows that best results are obtained when these are not the same.

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- While the scattering example shows that Halley's method can provide superior as well as faster reconstructions, there are problems (even those involving the detection of obstacles) for which this seems not to be the case. This may be due to difficulties in selecting an optimal level of regularisation or it may be inherent in the problem.

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