On Homogenization of Almost Periodic Nonlinear Parabolic Operators

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Abstract

In the present paper we prove an individual homogenization result for a class of almost periodic nonlinear parabolic operators.

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1 Introduction

In the present paper we consider the homogenization problem for nonlinear parabolic operators of the form

$$L_\varepsilon(u) = D_t u - div\left(a\left(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}, u, D_x u\right)\right) + a_0\left(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}, u, D_x u\right), \quad (1)$$

where the flux functions $a(y, \tau, \cdot, \cdot)$ and $a_0(y, \tau, \cdot, \cdot)$ are almost periodic in $(y, \tau) \in \mathbb{R}^{n+1}$. Here the parabolicity means that the leading elliptic part

$$-div\big(a(\cdot, \cdot, u, D_x u)\big)$$

is strictly monotone with respect to the gradient $D_x u$ (more precise assumptions are imposed below). Consequently, the entire elliptic term is pseudo monotone, but *not* monotone in general.

We are interested in the asymptotic behavior of L_ε as $\varepsilon \to 0$. G-convergence theory for parabolic operators guarantees that the limiting operator $\widehat{L}$ belongs to the same class of parabolic operators. G-convergence of nonlinear parabolic operators has been studied in [19]. To find the form of $\widehat{L}$ some assumptions on the nature of spatial and temporal heterogeneities of a and a_0 need to be imposed. In the periodic setting the homogenization of nonlinear parabolic equations is carried out in [19]. In [4], time homogenization of random nonlinear abstract parabolic equations has been studied. The homogenization of linear parabolic operators with almost periodic and random coefficients has been studied in [21, 20]. The homogenization of general nonlinear random parabolic operators is investigated by the authors in [11].

Also we mention several results on homogenization of nonlinear elliptic operators [2, 5, 12, 13, 16, 18]. Note that in [16, 18] general elliptic operators in divergence form are considered, including random homogenization, while articles [2, 5, 12, 13] are devoted to the case of monotone second order elliptic operators. As for general references in the field of homogenization, we refer to [1, 3, 6, 7, 14, 19].

We would like to point out that the general result of [11] is of statistical nature: homogenization takes place for almost all realizations of a random parabolic operator. As we shall see in Section 4 any almost periodic operator of the form (1) can be considered as a particular realization of certain random homogeneous operator. But this realization is not generic and, therefore, the result of [11] does not apply straightforwardly. Nevertheless, using almost periodicity one can pass from a generic realization to every particular realization and this is done in the proof of our main result in this paper.

Our motivation for considering homogenization of nonlinear parabolic equations comes from applications arisen in flow in porous media for both saturated and unsaturated media, though one encounter nonlinear parabolic equations in many different applications. We refer to [8, 9, 10] for numerical realization of parabolic homogenization in applied problems.

The paper is organized as follows. In the next section we collect some basic facts on G-convergence of parabolic operators. Section 3 contains the main result. In section 4 we present the proof of main theorem.

2 G-convergence of parabolic operators

Let $Q_0 \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $T > 0$, and $Q = Q_0 \times (0,T)$. On Q, we consider parabolic operators of the form

$$L(u) = D_t u - div\big(a(x,t,u,D_x u)\big) + a_0(x,t,u,D_x u). \tag{2}$$

We suppose that the functions $a : Q \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $a_0 : Q \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy the Carathéodory condition and the following assumptions:

(i) for every $\zeta = (\eta, \xi) \in \mathbb{R}^{n+1}$

$$\big|a(x,t,\eta,\xi)\big|^{p'} + \big|a_0(x,t,\eta,\xi)\big|^{p'} \le c_0|\zeta|^p + c \tag{3}$$

a. e. on Q, *where* $p > 1$, $c_0 > 0$ *and* $c \ge 0$;

(ii) for every $\zeta \in (\eta, \xi)$, $\zeta' = (\eta, \xi') \in \mathbb{R}^{n+1}$

$$\big[a(x,t,\eta,\xi) - a(x,t,\eta,\xi')\big] \cdot (\xi - \xi') \geq \kappa \big[h + |\zeta|^p + |\zeta'|^p\big]^{1-\beta/p} |\xi - \xi'|^\beta \quad (4)$$

a. e. on Q, *where* $\kappa > 0$, $\beta \geq \max(p, 2)$, *and* $h \geq 0$;

(iii) for every $\zeta = (\eta, \xi)$, $\zeta' = (\eta', \xi') \in \mathbb{R}^{n+1}$

$$\begin{aligned} \big|a(x,t,\eta,\xi) &- a(x,t,\eta',\xi')\big|^{p'} + \big|a_0(x,t,\eta,\xi) - a_0(x,t,\eta',\xi')\big|^{p'} \leq \\ &\leq \theta \Big[\big(h + |\zeta|^p + |\zeta'|^p\big)\, \nu\big(|\eta - \eta'|\big) + \\ &\quad + \big(h + |\zeta|^p + |\zeta'|^p\big)^{1-s/p} |\xi - \xi'|^s\Big], \end{aligned} \quad (5)$$

a. e. on Q, *where* $\theta > 0$, $0 < s \leq \min(p, p')$ *and* $\nu(r)$ *is a continuity modulus, i. e. a continuous function* $\nu : [0, +\infty) \to [0, +\infty)$ *such that* $\nu(0) = 0$, $\nu(r) > 0$ *if* $r > 0$ *and* $\nu(r) = 1$ *if* $r \geq 1$.

Here $p > 1$ is fixed and p' stands for the conjugate exponent, $(p')^{-1} + p^{-1} = 1$. In addition, we always assume that $p > 2n/(n+2)$.

Given $c_0, c, \kappa, h, \theta, \nu, s$ and β, we denote by $\Pi = \Pi(c_0, c, \kappa, h, \theta, \nu, s, \beta)$ the set of all operators of the form (2) satisfying *(i)*–*(iii)*.

For shortness, we employ the following notation for functional spaces. We denote by H the Hilbert space $L^2(Q_0)$ identified with its dual space and set

$$V = W_0^{1,p}(Q_0), \quad \overline{V} = W^{1,p}(Q_0), \quad V' = W^{-1,p'}(Q_0).$$

Also we introduce the following spaces

$$\begin{aligned} \mathcal{V} &= L^p(0,T;V), & \mathcal{V}' &= L^{p'}(0,T;V'), \\ \mathcal{W} &= \{u \in \mathcal{V} \,:\, D_t u \in \mathcal{V}'\}, & \mathcal{W}_0 &= \{u \in \mathcal{W} \,:\, u(0) = 0\}, \\ \overline{\mathcal{V}} &= L^p(0,T;\overline{V}), & \overline{\mathcal{W}} &= \{u \in \overline{\mathcal{V}} \,:\, D_t u \in \mathcal{V}'\}. \end{aligned}$$

Any operator $L \in \Pi$ acts from $\mathcal{W}_0$ into $\mathcal{V}'$.

Now we introduce the notion of G-convergence. For an operator L of the form (2) we set

$$L^1(u,v) = D_t u - div\big(a(x,t,v,D_x u)\big).$$

For every fixed $v \in \mathcal{V}$, the operator $u \mapsto L^1(u,v)$ acts from $\mathcal{W}_0$ into $\mathcal{V}'$ and satisfies the coerciveness and strict monotonicity conditions. Therefore, for every $f \in \mathcal{V}'$, $v \in \mathcal{V}$, the equation

$$L^1(u,v) = f$$

has a unique solution $u \in \mathcal{W}_0$ (see, e. g. [15]).

Now let $L_k \in \Pi$ be a sequence of parabolic operators, with fluxes a^k and a_0^k, and $L \in \Pi$ of the form (2). Given $u \in \mathcal{W}_0$ and $v \in \mathcal{V}$, we set

$$\begin{aligned} \Gamma^k(u,v) &= a^k(x,t,v,D_x u^k), \\ \Gamma_0^k(u,v) &= a_0^k(x,t,v,D_x u^k), \\ \Gamma(u,v) &= a(x,t,v,D_x u^k), \end{aligned}$$

and

$$\Gamma_0(u, v) = a_0(x, t, v, D_x u^k),$$

where $u_k \in \mathcal{W}_0$ is a unique solution of the equation

$$L_k^1(u_k, v) = L^1(u, v).$$

The sequence L_k is called *G-convergent* to L (in symbols, $L_k \xrightarrow{G} L$) if for every $v \in \mathcal{V}$ and $u \in \mathcal{W}_0$ we have that

$$\lim u_k = u$$

weakly in $\mathcal{W}_0$ and

$$\begin{aligned} \lim \Gamma^k(u, v) &= \gamma(u, v), \\ \lim \Gamma_0^k(u, v) &= \Gamma_0(u, v) \end{aligned}$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$, respectively, as $k \to \infty$. In the following analysis, $k \to \infty$ will be omitted. This notion was introduced in [19], where the term "strong G-convergence" was suggested. In this paper we abbreviate it to "G-convergence" because no other type of such convergence is used here.

The following G-compactness theorem is one of the main results of G-convergence theory (see [19], Theorem 4.1.1).

Theorem 2.1 *Let L_k be a sequence of parabolic operators of class Π. Then there exist an operator L of class Π, with possibly another values of parameters $c_0, c, \kappa, h, \theta, s,$ and ν, and a subsequence $L_{k'}$ that G-converges to L.*

We also need another type of convergence of parabolic operators, which could be named *coefficient-wise convergence*. Let L_1 and L_2 be two parabolic operators of the form (2), with flux functions (a^1, a_0^1) and (a^2, a_0^2), respectively. Set

$$d(L_1, L_2) = \sup \frac{\left|a^1(x,t,\eta,\xi) - a^2(x,t,\eta,\xi)\right| + \left|a_0^1(x,t,\eta,\xi) - a_0^2(x,t,\eta,\xi)\right|}{1 + |\eta|^{p-1} + |\xi|^{p-1}},$$

where the supremum is is taken over all $(x, t, \eta, \xi) \in Q_0 \times \mathbb{R}^{n+1}$. It is easy to check that d is a metric on the set of parabolic operators (remind that $p > 2n/(n+2)$ is fixed). The following result is borrowed from [19] (see Proposition 4.1.2).

Proposition 2.1 *Assume that $d(L_k^l, L_k) \to 0$ uniformly with respect to k and $d(L^l, L) \to 0$ as $l \to \infty$. If $L_k^l \xrightarrow{G} L^l$ for every l, then $L_k \xrightarrow{G} L$.*

3 Main result

Consider the operator L_ε defined by (1). We suppose that the flux functions $a(y, \tau, \eta, \xi)$ and $a_0(y, \tau, \eta, \xi)$ satisfy assumptions (i)–(iii), with $(x, t) \in Q$ replaced by $(y, \tau) \in \mathbb{R}^{n+1}$. Remind that $p > 2n/(n+2)$ is fixed. We impose the following almost periodicity assumption:

(iv) The functions

$$\frac{a(y,\tau,\eta,\xi)}{1+|\eta|^{p-1}+|\xi|^{p-1}}$$

and

$$\frac{a_0(y,\tau,\eta,\xi)}{1+|\eta|^{p-1}+|\xi|^{p-1}}$$

are almost periodic in the sense of Bohr uniformly with respect to $(\eta,\xi)\in\mathbb{R}^{n+1}$.

For the following theorem, we assume that $\alpha\geq 0$, $\beta\geq 0$ and $\alpha^2+\beta^2\neq 0$.

Theorem 3.1 *Suppose that assumptions (i)–(iii), with* $(x,t)\in Q$ *replaced by* $(y,\tau)\in\mathbb{R}^{n+1}$*, and (iv) are satisfied. Then there exists an operator*

$$\widehat{L}u = D_t u - div\,\widehat{a}(x,t,u,D_x u) + \widehat{a}_0(x,t,u,D_x u)$$

such that, for every Lipschitz domain $Q_0\subset\mathbb{R}^n$ *and every* $T>0$, $L_\varepsilon \stackrel{G}{\to} \widehat{L}$ *on* Q. *If* $\alpha>0$ *and* $\beta>0$*, then the flux functions* $\widehat{a}$ *and* $\widehat{a}_0$ *are independent of* x *and* t. *If* $\alpha=0$, $\beta=0$*, then* $\widehat{a}$ *and* $\widehat{a}_0$ *are almost periodic in the sense of (iv) and independent of* x. *If* $\alpha>0$, $\beta=0$*, then* $\widehat{a}$ *and* $\widehat{a}_0$ *are almost periodic and independent of* t.

Precise description of operator $\widehat{L}$ can be found in [11], where this operator appears in random homogenization. Actually, Theorem 3.1 shows that the same construction of homogenized operator serves individual almost periodic parabolic homogenization. Here we only mention that in case $\alpha>0$, $\beta=0$ (time homogenization)

$$\widehat{a}(x,\eta,\xi) = \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau} a(x,t,\eta,\xi)\,dt, \tag{6}$$

$$\widehat{a}_0(x,\eta,\xi) = \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau} a_0(x,t,\eta,\xi)\,dt, \tag{7}$$

while in case $\alpha=0$, $\beta>0$ (spatial homogenization) it is sufficient to homogenize, according to [19], the elliptic part of the operator L for every $t\in\mathbb{R}$.

4 Proof of main result

To prove individual homogenization, first we include the operators L_ε into a family of operators $L_{\omega,\varepsilon}$, $\omega\in\Omega$, parametrized by a certain probability space Ω so that $L_{\omega,\varepsilon}$ form random homogeneous parabolic operators. Next, we apply the homogenization result of [11] to show that $L_{\omega,\varepsilon}$ possesses homogenization for almost all $\omega\in\Omega$. Finally, we use Proposition 2.1 to prove that the last statement holds for all $\omega\in\Omega$.

We take Ω to be the Bohr compactification $\mathbb{R}^{n+1}_B$ of $\mathbb{R}^{n+1}$ [17]. Recall that this is a compact abelian group and there is a continuous dense embedding

$\mathbb{R}^{n+1} \subset \mathbb{R}_B^{n+1}$ of abelian groups. Endowed with the normalized Haar measure, $\Omega = \mathbb{R}_B^{n+1}$ is a probability space. Moreover, every Bohr almost periodic function on $\mathbb{R}^{n+1}$ extends uniquely to a continuous function on $\mathbb{R}_B^{n+1}$.

Thus, for every $(\eta, \xi) \in \mathbb{R}^{n+1}$ we can extend $a(y, \tau, \eta, \xi)$ and $a_0(y, \tau, \eta, \xi)$ to continuous functions on $\mathbb{R}_B^{n+1}$. We denote these extensions by $a(\omega, \eta, \xi)$ and $a_0(\omega, \eta, \xi)$, respectively. Assumption (iv) imply immediately that $a(\omega, \eta, \xi)$ and $a_0(\omega, \eta, \xi)$ are continuous functions on $\mathbb{R}_B^{n+1} \times \mathbb{R}^{n+1}$. Moreover, assumptions (i)–(iii) are satisfied for these extensions, with (x, t) replaced by ω. Now we define $L_{\omega,\varepsilon}$ as follows

$$L_{\omega,\varepsilon}(u) = D_t u - div\, a\left(\omega + \left(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}\right), u, D_x u\right) + a_0\left(\omega + \left(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}\right), u, D_x u\right).$$

Obviously $L_\varepsilon = L_{0,\varepsilon}$.

Let us consider the case $\alpha > 0$, $\beta > 0$. Because of the results obtained in [11], there exist a subset $\Omega_0 \subset \Omega = \mathbb{R}_B^{n+1}$ of measure 1 and an operator

$$\widehat{L}u = D_t u - div\, \widehat{a}(u, D_x u) + \widehat{a}_0(u, D_x u)$$

such that, for every Lipschitz domain Q_0 and every $T > 0$, $L_{\omega,\varepsilon} \xrightarrow{G} \widehat{L}$ on $Q = Q_0 \times T$ for all $\omega \in \Omega_0$.

Since Ω_0 is a subset of full measure, it is also a dense subset of $\mathbb{R}_B^{n+1}$. Hence, there exists a sequence (more precisely, a net) $\omega_l \in \Omega_0$ such that $\omega_l \to 0$ in $\mathbb{R}_B^{n+1}$. Moreover, by Theorem 2.1, for every sequence $\varepsilon' \to 0$ there exists a subsequence ε_k of ε' such that $L_k = L_{\varepsilon_k} = L_{0,\varepsilon_k} \xrightarrow{G} \widetilde{L}$ on Q for some parabolic operator $\widetilde{L}$ of class Π. Now we are in the situation of Proposition 2.1. Hence, $\widetilde{L} = \widehat{L}$. In particular, the passage to a subsequence ε_k is superfluous and we obtain that $L_\varepsilon \xrightarrow{G} \widehat{L}$. Certainly, this holds for every $\omega \in \mathbb{R}_B^{n+1}$, not only for $\omega = 0$.

Case $\alpha > 0$, $\beta = 0$ is similar. The only difference is that now $\widehat{L}$ depends on ω. More precisely, the flux functions $\widehat{a}$ and $\widehat{a}_0$ are of the form $\widehat{a}(\omega, \eta, \xi)$ and $\widehat{a}_0(\omega, \eta, \xi)$. Actually, due to the construction of $\widehat{L}$ [11] (see also (6) and (7)) the dependence on ω is of the following form. The Bohr compactification $\mathbb{R}_B^{n+1}$ splits into the direct product $\mathbb{R}_B^n \times \mathbb{R}_B$ and for every point $\omega \in \mathbb{R}_B^{n+1}$ we can write $\omega = (\omega', \omega'')$, where $\omega' \in \mathbb{R}_B^n$ and $\omega'' \in \mathbb{R}_B$. The flux functions $\widehat{a}$ and $\widehat{a}_0$ depend only on ω'. Moreover, in the case under consideration

$$\widehat{a}(\omega', \eta, \xi) = \int_{\mathbb{R}_B} a\big((\omega', \omega''), \eta, \xi\big)\, d\omega''$$

and similarly for $\widehat{a}_0$. This shows that the families of functions

$$\frac{\widehat{a}(\omega', \eta, \xi)}{1 + |\eta|^{p-1} + |\xi|^{p-1}}$$

and

$$\frac{\widehat{a}_0(\omega', \eta, \xi)}{1 + |\eta|^{p-1} + |\xi|^{p-1}},$$

with $(\eta, \xi) \in \mathbb{R}^{n+1}$ considered as a parameter, are equicontinuous in $\omega' \in \mathbb{R}^n_B$. This is sufficient to apply Proposition 2.1 exactly as in the case $\alpha > 0$, $\beta > 0$.

In the case of $\alpha = 0$, $\beta > 0$, the assumption (iv) implies the regularity assumption of Theorem 4.1.4 of [19] and we obtain the desired result.

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References

[1] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam, 1978.

[2] A. Braides, V. Chiadò Piat , and A. Defranceschi, *Homogenization of almost periodic monotone operators*, Ann. Inst. H. Poincaré, Anal. non Lin. **9** (1992), 399–432.

[3] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Clarendon Press, Oxford Univ. Press, New York, 1998.

[4] G. Bruno, A. Pankov, and T. Pankova, *Time averaging for random nonlinear abstract parabolic equations*, Abstr. Appl. Anal. **5** (2000), 1–11.

[5] V. Chiadò Piat, G. Dal Maso, and A. Defranceschi, *G-convergence of monotone operators*, Ann. Inst. H. Poincaré, Anal. non Lin. **7** (1990), 123–160.

[6] D. Cioranescu and P. Donato, *An Introduction to Homogenization*, Clarendon Press, Oxford Univ. Press, New York, 1999.

[7] G. Dal Maso, *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[8] Y. Efendiev and A. Pankov, *Numerical homogenization of monotone elliptic operators*, SIAM Multiscale Model. Simul. **1** (2003), 62–79.

[9] Y. Efendiev and A. Pankov, *Numerical homogenization of nonlinear random parabolic operators*, SIAM Multiscale Model. Simul. **2** (2004), 237–268.

[10] Y. Efendiev and A. Pankov, *Numerical homogenization and correctors for nonlinear elliptic equations*, SIAM J. Appl. Math. **65** (2004), 43–68.

[11] Y. Efendiev and A. Pankov, *Homogenization of nonlinear random parabolic operators*, Adv. Differ. Equat. (submitted).

[12] N. Fusco and G. Moscariello, *Further results on the homogenization of quasilinear operators*, Ric Mat. **35** (1986), 231–246.

[13] N. Fusco and G. Moscariello, *On the homogenization of quasilinear divergence structure operators*, Ann. Mat. Pura. Appl. **146** (1987), 1–13.

[14] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, New York,1994.

[15] J. L. Lions, *Quelques Méthodes de Résoluton des Problèmes aux Limites Non Linéairies*, Dunod, Paris, 1969.

[16] A. Pankov, *Homogenization and G-convergence of nonlinear elliptic operators in divergence form*, Dokl. AN SSSR **278** (1984), 37–41 (in Russian).

[17] A. Pankov, *Bounded and Almost periodic Solutions of Nonlinear Operator Differential Equations*, Kluwer, Dordrecht, 1990.

[18] A. Pankov, *Strong G-convergence of nonlinear elliptic operators and homogenization*, Constanin Carathéodory — An International Tribute (ed. Th. Rassias), World Sci. Publ., Singapore, 1991, 1075–1099.

[19] A. Pankov, *G-convergence and Homogenization of Nonlinear Partial Differential Operators*, Kluwer Academic Publishers, Dordrecht, 1997.

[20] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭnik, *Averaging of parabolic operators*, Trudy Moskov. Mat. Obshch. **45** (1982), 182–236 (in Rusian).

[21] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭnik, *Averaging of parabolic operators with almost periodic coefficients*, Mat. Sb. (N.S.) **117**(159) (1982), 69–85 (in Russian).