1	/ 8	5	/16	8	/10
2	/ 8	6	/12	9	/12
3	/ 8	7	/ 8	10	/13
4	/10	Total			/105

Name______________________________________

MATH 172 Honors   Exam 3   Spring 2024

Section 200   Solutions   P. Yasskin

Points indicated. Part credit possible. Show all work.

1. (8 points) Compute the following limits.

a. (2 pts) $\lim_{n\to\infty} \dfrac{n^2 - 3n + 2}{n + 1}$

Solution: $\lim_{n\to\infty} \dfrac{n^2 - 3n + 2}{n + 1} = \lim_{n\to\infty} \dfrac{n - 3 + \frac{2}{n}}{1 + \frac{1}{n}} = \infty$

b. (2 pts) $\lim_{n\to\infty} \sqrt[n]{3n + 1}$

Solution: $\lim_{n\to\infty} \sqrt[n]{3n + 1} = 1$ because the n^{th}-root of any polynomial is 1.

c. (4 pts) $\lim_{n\to\infty} \left(2 + \frac{3}{n}\right)^{4n}$

Solution: Let $L = \lim_{n\to\infty} \left(2 + \frac{3}{n}\right)^{4n}$. Then

$$\ln L = \lim_{n\to\infty} \ln\left(2 + \frac{3}{n}\right)^{4n} = \lim_{n\to\infty} 4n \ln\left(2 + \frac{3}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(2 + \frac{3}{n}\right)}{\frac{1}{4n}} \overset{l'H}{=} \lim_{n\to\infty} \frac{\frac{-\frac{3}{n^2}}{2 + \frac{3}{n}}}{-\frac{1}{4n^2}} = \lim_{n\to\infty} \frac{\frac{3}{n^2}}{2 + \frac{3}{n}} \frac{4n^2}{1} = 6$$

$L = e^6$

2. (8 points) Find a power series for $\arctan(x) = \int_0^x \frac{1}{1 + t^2}\, dt$ centered at $x = 0$.
Hint: Write a series for each of the following:

Solution:

a. $\dfrac{1}{1 - r} = \underline{\sum_{n=0}^{\infty} r^n}$

b. $\dfrac{1}{1 + t^2} = \underline{\sum_{n=0}^{\infty} (-t^2)^n} = \underline{\sum_{n=0}^{\infty} (-1)^n t^{2n}}$

c. $\int \dfrac{1}{1 + t^2}\, dt = \underline{\sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+1}}{2n + 1}} + C$

d. $\int_0^x \dfrac{1}{1 + t^2}\, dt = \underline{\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n + 1}}$

3. (8 points) Compute the sum of each series. Simplify each to a single rational number.

a. (4 pts) $S = \sum_{n=1}^{\infty} \frac{3^n + 4^n}{12^n}$ HINT: First split the terms into the sum of two fractions.

Solution: $\sum_{n=1}^{\infty} \frac{3^n + 4^n}{12^n} = \sum_{n=1}^{\infty} \frac{3^n}{12^n} + \sum_{n=1}^{\infty} \frac{4^n}{12^n} = \sum_{n=1}^{\infty} \frac{1}{4^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$

Each sum is geometric. The ratios $\frac{1}{4}$ and $\frac{1}{3}$ are both less than 1. So

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{12^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} + \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{4-1} + \frac{1}{3-1} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

b. (4 pts) $S = \sum_{n=2}^{\infty} \frac{1}{n(n+1)}$ HINT: First find the partial fraction expansion for $\frac{1}{n(n+1)}$.

Solution: $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ $\quad 1 = A(n+1) + Bn$

$n = 0 \quad \Rightarrow \quad A = 1 \qquad n = -1 \quad \Rightarrow B = -1$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \qquad S = \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$ This is telescoping.

$$S_k = \sum_{n=2}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{2} - \frac{1}{k+1}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(\frac{1}{2} - \frac{1}{k+1} \right) = \frac{1}{2}$$

4. (10 points) An egg is taken out of the refrigerator at $40°F$ and immediately put in a pot of boiling water at $212°F$. After 1 minute, the temperature has risen to $126°F$. How long will it take for the egg to reach $169°F$?
NOTE: These numbers have nothing to do with the real world.
They are chosen to make two fractions simplify to $\frac{1}{2}$ and $\frac{1}{4}$.

Solution: Newton's Law of Heating says $\frac{dT}{dt} = -k(T - T_A)$ where T_A is the ambient temperature.

The solution is $T = T_A + (T_o - T_A)e^{-kt}$ where T_o is the initial temperature.

Here, $T_A = 212$ and $T_o = 40$. To find k, we plug in $t = 1$ and $T = 126$.

$126 = 212 + (40 - 212)e^{-k1} \qquad e^{-k} = \frac{126 - 212}{40 - 212} = \frac{86}{172} = \frac{1}{2} \qquad -k = \ln\frac{1}{2} \qquad k = \ln 2$

So $T = 212 + (40 - 212)e^{-t\ln 2} = 212 - 172 \cdot 2^{-t}$ We need to solve $169 = 212 - 172 \cdot 2^{-t}$.

$172 \cdot 2^{-t} = 212 - 169 = 43 \qquad 2^{-t} = \frac{43}{172} = \frac{1}{4} \qquad t = 2$

5. (16 points) A sequence is defined by $a_1 = 4$ and $a_{n+1} = 2\sqrt{a_n} + 3$.

a. (2 pts) Find a_2 and a_3.

Solution: $a_2 = 2\sqrt{a_1} + 3 = 2\sqrt{4} + 3 = 7 \qquad a_3 = 2\sqrt{a_2} + 3 = 2\sqrt{7} + 3$

b. (4 pts) Assuming the limit exists, find the possible values of the limit.

Solution: Assuming the limit exists, let $L = \lim_{n\to\infty} a_n$. Then $\lim_{n\to\infty} a_{n+1} = L$ also.
The recursion formula says
$L = 2\sqrt{L} + 3 \qquad (L-3)^2 = 4L \qquad L^2 - 6L + 9 = 4L \qquad L^2 - 10L + 9 = 0$
$(L-1)(L-9) = 0 \qquad L = 1, 9$ The limit must be 1 or 9.

c. (4 pts) Prove the sequence is increasing or decreasing.

Solution: We will show a_n is increasing, i.e. $a_n < a_{n+1}$.
Init: $a_1 = 4 < a_2 = 7$
Induc: Assume $a_k < a_{k+1}$. Prove $a_{k+1} < a_{k+2}$.
Proof: $a_k < a_{k+1} \qquad \sqrt{a_k} < \sqrt{a_{k+1}} \qquad 2\sqrt{a_k} + 3 < 2\sqrt{a_{k+1}} + 3 \qquad a_{k+1} < a_{k+2}$ Proved.

d. (4 pts) Prove the sequence is bounded above (if increasing) or below (if decreasing).

Solution: We will show a_n is bounded above by 9, i.e. $a_n < 9$.
Init: $a_1 = 4 < 9$
Induc: Assume $a_k < 9$. Prove $a_{k+1} < 9$.
Proof: $a_k < 9 \qquad \sqrt{a_k} < 3 \qquad 2\sqrt{a_k} < 6 \qquad 2\sqrt{a_k} + 3 < 9 \qquad a_{k+1} < 9$ Proved.

e. (2 pts) State the theorem that shows the sequence converges and state the limit.

Solution: Since a_n is increasing and bounded above by 9,
the Bounded-Monotonic Sequence Theorem says it converges.
$\lim_{n\to\infty} a_n = 9$.

6. (12 points) Determine whether each series is convergent or divergent.
Name each Convergence Test(s) you use and check their assumptions. Use sentences!

a. (4 pts) $\displaystyle\sum_{n=0}^{\infty} \frac{5}{n^3 + \sqrt[3]{n}}$

Solution: $\displaystyle\frac{5}{n^3 + \sqrt[3]{n}} < \frac{5}{n^3}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{5}{n^3}$ converges because it is a p-series with $p = 3 > 1$.

So $\displaystyle\sum_{n=0}^{\infty} \frac{5}{n^3 + \sqrt[3]{n}}$ also converges by the Simple Comparison Test.

b. (4 pts) $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{(1+e^n)^2}$

Solution: We apply the Integal Test. The function $\displaystyle\frac{e^n}{(1+e^n)^2}$ is positive and decreasing and

$$\int_0^{\infty} \frac{e^n}{(1+e^n)^2}\,dn = \left[\frac{-1}{1+e^n}\right]_0^{\infty} = \frac{-1}{1+e^{\infty}} - \frac{-1}{1+e^0} = 0 + \frac{1}{2} = \frac{1}{2}$$

So the series converges by the Integal Test. OR Use a Simple Comparison with $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$.

c. (4 pts) $\displaystyle\sum_{n=2}^{\infty} \frac{n + \sqrt{n}}{n^2 + \sqrt{n}}$

Solution: We want to compare to $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n}$ which is the divergent harmonic series.

However, we cannot tell if $\displaystyle\frac{n + \sqrt{n}}{n^2 + \sqrt{n}}$ is smaller or larger than $\displaystyle\frac{1}{n}$.

So we cannot use the Simple Comparison Test. We use the Limit Comparison Test.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + \sqrt{n}}{n^2 + \sqrt{n}} \frac{n}{1} = \lim_{n\to\infty} \frac{n^2 + n\sqrt{n}}{n^2 + \sqrt{n}} = 1$$

So the original series also diverges.

7. (8 points) The series $\displaystyle S = \sum_{n=0}^{\infty} \frac{24n^2}{(n^3 + 1000)^3}$ is convergent by the Integral Test.

If the series is approximated by its 10^{th} partial sum $\displaystyle S_{10} = \sum_{n=0}^{10} \frac{24n^2}{(n^3 + 1000)^3}$,

find a bound on the error in this approximation: $E_{10} = S - S_{10}$.

Solution: $$E_{10} = \sum_{n=11}^{\infty} \frac{24n^2}{(n^3 + 1000)^3} < \int_{10}^{\infty} \frac{24n^2}{(n^3 + 1000)^3}\,dn = \left[\frac{-4}{(n^3 + 1000)^2}\right]_{10}^{\infty} = 0 - \frac{-4}{(10^3 + 1000)^2}$$
$$= \frac{4}{(2000)^2} = 10^{-6}$$

8. (10 points) The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ is: (Explain all reasoning. Circle your answer.)

a. Absolutely Convergent
b. Absolutely Divergent
c. Conditionally Convergent
d. Conditionally Divergent
e. Divergent

Solution: The Related Absolute Series is $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$. We apply the Limit Comparison Test comparing to $\sum_{n=0}^{\infty} \frac{1}{n}$ which is a divergent harmonic series.

$L = \lim_{n\to\infty} \frac{a_b}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \frac{n}{1} = 1$ So the absolute series is also divergent.

We apply the Alternating Series Test to the original series.

The $(-1)^n$ says it alternates. $\frac{1}{\sqrt{n^2+1}}$ decreases and $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$.

So the original series converges and so is Conditionally Convergent.

9. (12 points) A bucket contains 50 gal of salt water containing 5 lb of salt.

Salt water with a concentration of $0.2 \frac{\text{lb}}{\text{gal}}$ is added at the rate $5 \frac{\text{gal}}{\text{min}}$.

The water is kept well mixed and drained at $5 \frac{\text{gal}}{\text{min}}$.

How much salt is in the bucket after 10 min?

Solution: Let $S(t)$ be the amount of salt in the bucket at time t.

The differential equation is $\frac{dS}{dt} = 0.2 \frac{\text{lb}}{\text{gal}} 5 \frac{\text{gal}}{\text{min}} - \frac{S \text{ lb}}{50 \text{ gal}} 5 \frac{\text{gal}}{\text{min}}$

It's linear. The standard form is: $\frac{dS}{dt} + \frac{1}{10} S = 1$ We identify $P = \frac{1}{10}$.

The integrating factor is $I = \exp\left(\int P\,dt\right) = e^{t/10}$.

We multiply the standard form and integrate:

$$e^{t/10}\frac{dS}{dt} - \frac{1}{10}e^{t/10}S = e^{t/10} \qquad \frac{d}{dt}(e^{t/10}S) = e^{t/10} \qquad e^{t/10}S = \int e^{t/10}\,dt = 10e^{t/10} + C$$

The initial condition is $S(0) = 5$, or $t = 0$ and $S = 5$. So $e^0 5 = 10e^0 + C$ $C = -5$

So the solution is $S = 10 - 5e^{-t/10}$. $S(10) = 10 - 5e^{-1}$

10. (13 points) Find the Interval of Convergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}\,3^n}(x-2)^n$.

Follow these steps. Be sure to name any Convergence Tests you use.

a. (4 pts) Find the Radius of Convergence.

Solution: We apply the Ratio Test. $|a_n| = \frac{1}{\sqrt{n}\,3^n}|x-2|^n \qquad |a_{n+1}| = \frac{1}{\sqrt{n+1}\,3^{n+1}}|x-2|^{n+1}$

$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{\sqrt{n+1}\,3^{n+1}} \frac{\sqrt{n}\,3^n}{|x-2|^n} = \frac{|x-2|}{3} < 1 \qquad |x-2| < 3 \qquad R = 3$

Open interval of convergence is $(2-3, 2+3) = (-1, 5)$

b. (4 pts) Check convergence at the left endpoint.

Be sure to name and check out any Convergence Tests you use.

$x =$ ______________

Solution: $x = -1 \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}\,3^n}(-3)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ $\quad$ p-series with $p = \frac{1}{2} < 1$ $\quad$ divergent

c. (4 pts) Check convergence at the right endpoint.

Be sure to name and check out any Convergence Tests you use.

$x =$ ______________

Solution: $x = 5 \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}\,3^n}(3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\quad$ Alt Ser, decr. $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$ $\quad$ convergent

d. (1 pts) State the Interval of Convergence.

Solution: $(-1, 5]$