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MATH 221 Exam 3, Version B Spring 2024

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7	/20
8	/15
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Total	/104

Multiple Choice: (9 points each. No part credit.)
Circle you answers here and bubble on the Scantron.
Show your work, in case I give some part credit.

1. Compute $\int_0^2 \int_p^{2p} pq\,dq\,dp$.

a. 24
b. 12
c. 8
d. 6 Correct
e. 4

Solution: $\int_0^2 \int_p^{2p} pq\,dq\,dp = \int_0^2 \left[p\frac{q^2}{2}\right]_{q=p}^{2p} dp = \int_0^2 \left(p\frac{4p^2}{2}\right) - \left(p\frac{p^2}{2}\right) dp = \frac{3}{2}\int_0^2 p^3\,dp = \frac{3}{2}\left[\frac{p^4}{4}\right]_0^2 = 6$

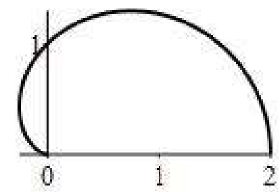
2. Approximate the integral $\int_2^6 \int_1^5 (x^2+y)\,dy\,dx$ by a Riemann sum using 4 squares and evaluating at the center of each square.

a. 80
b. 160
c. $\frac{488}{3}$
d. $\frac{976}{3}$
e. 320 Correct

Solution: The squares have width and height $\Delta x = 2$ and $\Delta y = 2$ and so area $\Delta A = \Delta x \Delta y = 4$. The centers are $(3,2)$, $(3,4)$, $(5,2)$, $(5,4)$. The function is $f = x^2 + y$. The function values are: $f(3,2) = 11$, $f(3,4) = 13$, $f(5,2) = 27$, $f(5,4) = 29$ So the Riemann sum approximation is:

$$\int_2^6 \int_1^5 (x^2+y)\,dy\,dx = \sum_{i=1}^{4} f(x_i,y_i)\Delta A = (11+13+27+29)4 = 320$$

3. Find the area of the interior of the upper half of the cardioid $r = 1 + \cos\theta$.



a. $A = \frac{3\pi}{2}$

b. $A = \frac{3\pi}{4}$ Correct

c. $A = \frac{2}{3\pi}$

d. $A = 2\pi$

e. $A = \pi$

Solution: In polar coordinates, $y = r\sin\theta$. The area is:

$$A = \int_0^\pi \int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^\pi \left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta = \frac{1}{2}\int_0^\pi (1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^\pi \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = \frac{3}{4}\pi$$

4. Find the average value of the function $f(x,y) = y$ on the interior of the upper half of the cardioid $r = 1 + \cos\theta$ as shown in problem 3.

a. $f_{\text{ave}} = \frac{16}{9\pi}$ Correct

b. $f_{\text{ave}} = \frac{9\pi}{16}$

c. $f_{\text{ave}} = \frac{4}{9\pi}$

d. $f_{\text{ave}} = \frac{4}{3}$

e. $f_{\text{ave}} = \frac{3}{4}$

Solution: The area was found in the previous problem to be $A = \frac{3}{4}\pi$.

In polar coordinates, $y = r\sin\theta$. So $f = y = r\sin\theta$. The integral of f is:

$$\iint f\,dA = \int_0^\pi \int_0^{1+\cos\theta} r\sin\theta\, r\,dr\,d\theta = \int_0^\pi \sin\theta\left[\frac{r^3}{3}\right]_0^{1+\cos\theta} d\theta = \frac{1}{3}\int_0^\pi (1+\cos\theta)^3 \sin\theta\,d\theta = -\frac{1}{3}\int u^3\,du$$

$$= -\frac{u^4}{12} = -\left[\frac{(1+\cos\theta)^4}{12}\right]_0^\pi = -\left(0 - \frac{2^4}{12}\right) = \frac{4}{3}$$

So the average is $f_{\text{ave}} = \frac{1}{A}\iint f\,dA = \frac{4}{3\pi}\frac{4}{3} = \frac{16}{9\pi}$

5. Find the volume of the apple given in spherical coordinates by $\rho = 1 - \cos\phi$.

a. $2\pi^2$
b. $\frac{4\pi}{3}$
c. $\frac{8\pi}{3}$ Correct
d. $\frac{4\pi}{5}$
e. $\frac{8\pi}{5}$

Solution: $V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \int_0^{\pi}\left[\frac{\rho^3}{3}\right]_0^{1-\cos\phi} \sin\phi\, d\phi$

$$= \frac{2\pi}{3}\int_0^{\pi}(1-\cos\phi)^3 \sin\phi\, d\phi = \frac{2\pi}{3}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi} = \frac{\pi}{6}(2^4 - 0) = \frac{8\pi}{3}$$

6. Find the surface area of the parametric surface parametrized by $\vec{R}(u,v) = \langle u+v, u-v, 2u+2v\rangle$ for $0 \le u \le 2$ and $0 \le v \le 3$.

a. $12\sqrt{5}$ Correct
b. $6\sqrt{5}$
c. $2\sqrt{5}$
d. 6
e. 36

Solution:
$$\begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = \langle & 1, & 1, & 2\rangle \\ \vec{e}_\theta = \langle & 1, & -1, & 2\rangle \end{matrix}$$

$\vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(2+2) - \hat{\jmath}(2-2) + \hat{k}(-1-1) = \langle 4, 0, -2\rangle$

$|\vec{N}| = \sqrt{16+4} = 2\sqrt{5}$ $\quad A = \iint 1\,dS = \iint |\vec{N}|\,du\,dv = \int_0^3\int_0^2 2\sqrt{5}\,du\,dv = 2\sqrt{5}\,(2)(3) = 12\sqrt{5}$

Work Out: (Points indicated. Part credit possible. Show all work.)

7. (20 points) Consider the region between the y-axis and the parabola $x = 9 - y^2$.

a. Find the mass of the region, if the surface density is $\delta = y^2$.

Solution: $M = \iint \delta\, dA = \int_{-3}^{3}\int_{0}^{9-y^2} y^2 dx\, dy = \int_{-3}^{3} y^2\Big[x\Big]_0^{9-y^2} dy = \int_{-3}^{3} y^2(9-y^2)\, dy$

$= \int_{-3}^{3}(9y^2 - y^4)\, dy = \left[\frac{9y^3}{3} - \frac{y^5}{5}\right]_{-3}^{3} = 2\left(\frac{9\cdot 3^3}{3} - \frac{3^5}{5}\right)$

$= 2\cdot 3^5\left(\frac{1}{3} - \frac{1}{5}\right) = 2\cdot 3^5 \frac{2}{15} = \frac{4\cdot 3^4}{5}$

b. Find the center of mass of the region, if the surface density is $\delta = y^2$.

Write your answer as an ordered pair.

Solution: From the previous part the mass is $M = \frac{4\cdot 3^4}{5}$.

By symmetry $\bar{y} = 0$. The x-moment is

$M_x = \iint x\delta\, dA = \int_{-3}^{3}\int_{0}^{9-y^2} xy^2 dx\, dy = \int_{-3}^{3} y^2\left[\frac{x^2}{2}\right]_0^{9-y^2} dy = \frac{1}{2}\int_{-3}^{3} y^2(9-y^2)^2\, dy$

$= \frac{1}{2}\int_{-3}^{3} y^2(81 - 18y^2 + y^4)\, dy = \frac{1}{2}\int_{-3}^{3}(81y^2 - 18y^4 + y^6)\, dy = \frac{1}{2}\left[27y^3 - \frac{18y^5}{5} + \frac{y^7}{7}\right]_{-3}^{3}$

$= \left(27\cdot 3^3 - \frac{18\cdot 3^5}{5} + \frac{3^7}{7}\right) = 3^6\left(1 - \frac{6}{5} + \frac{3}{7}\right) = 3^6\frac{35 - 42 + 15}{35} = \frac{8\cdot 3^6}{35}$

$\bar{x} = \frac{M_x}{M} = \frac{8\cdot 3^6}{35}\frac{5}{4\cdot 3^4} = \frac{2\cdot 9}{7} = \frac{18}{7}$

So the center of mass is $(\bar{x},\bar{y}) = \left(\frac{18}{7}, 0\right)$.

8. (15 points) Compute $\iiint_P \vec{\nabla}\cdot\vec{F}\, dV$ for $\vec{F} = \langle xz, yz, z^2\rangle$ over the solid above the paraboloid $z = x^2 + y^2$ below the plane $z = 4$.

Solution: $\vec{\nabla}\cdot\vec{F} = z + z + 2z = 4z$ bottom surface is $z = x^2 + y^2 = r^2$ $dV = r\, dr\, d\theta\, dz$

$\iiint_P \vec{\nabla}\cdot\vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 4z\, r\, dz\, dr\, d\theta = 2\pi\int_0^2\Big[2z^2\Big]_{r^2}^4 r\, dr = 4\pi\int_0^2(16 - r^4)r\, dr$

$= 4\pi\left[\frac{16r^2}{2} - \frac{r^6}{6}\right]_0^2 = 4\pi\left(\frac{64}{2} - \frac{64}{6}\right) = 256\pi\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{256\pi}{3}$

9. (15 points) Compute $\iint_P \vec\nabla \times \vec F \cdot d\vec S$ for $\vec F = \langle -yz, xz, z^2 \rangle$

over the parabolic surface $z = x^2 + y^2$ below $z = 4$

oriented down and out. It may be parametrized by

$$\vec R(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$$

Solution: $\vec\nabla \times \vec F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = \hat\imath(0 - x) - \hat\jmath(0 - -y) + \hat k(z - -z) = \langle -x, -y, 2z \rangle$

On the surface $\vec\nabla \times \vec F\big|_{\vec R} = \langle -r\cos\theta, -r\sin\theta, 2r^2 \rangle$

$$\begin{matrix} & \hat\imath & \hat\jmath & \hat k \\ \vec e_r = & \langle \cos\theta, & \sin\theta, & 2r \rangle \\ \vec e_\theta = & \langle -r\sin\theta, & r\cos\theta, & 0 \rangle \end{matrix}$$

$\vec N = \vec e_u \times \vec e_v = \hat\imath(-2r^2\cos\theta) - \hat\jmath(2r^2\sin\theta) + \hat k(r\cos^2\theta + r\sin^2\theta)$

$= \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$ This is up and in. Need down and out.

Reverse $\vec N = \langle 2r^2\cos\theta, 2r^2\sin\theta, -r \rangle$

$\vec\nabla \times \vec F\big|_{\vec R} \cdot \vec N = -2r^3\cos^2\theta - 2r^3\sin^2\theta - 2r^3 = -4r^3$

$$\iint_P \vec\nabla \times \vec F \cdot d\vec S = \iint_P \vec\nabla \times \vec F\big|_{\vec R} \cdot \vec N\, dr\, d\theta = \int_0^{2\pi}\int_0^2 -4r^3\, dr\, d\theta = -2\pi\left[r^4\right]_0^2 = -32\pi$$