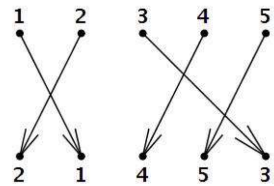
Definition and Properties of Permutations

A **permutation** is a rearrangement of things in a set where order matters. We here discuss the permutations of the set $\mathbb{Z}_n = \{1, 2, \cdots, n\}$. So a permutation of $\mathbb{Z}_n$ is a function

$$p : \mathbb{Z}_n \to \mathbb{Z}_n : i \mapsto p_i$$

where $\{p_1, p_2, \cdots p_n\} = \mathbb{Z}_n$. In other words, p is $1-1$ (injective) and onto (surjective). We usually write a permution as an order n-tuple: $p = (p_1, p_2, \cdots p_n)$. For example, if we consider permutations of $\mathbb{Z}_5$, then the permutation $(2,1,4,5,3)$ is the function



or more briefly, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$

Property:

1. There are $n!$ permutations of $\mathbb{Z}_n$.

A **transposition** is a permutation in which exactly 2 numbers are interchanged. An **adjacent transposition** is a transposition in which the 2 numbers are consecutive. For example, if we consider permutations of $\mathbb{Z}_5$,

$(3,2,4,1,5)$ is a permutation, $(4,2,3,1,5)$ is a transposition and $(1,3,2,4,5)$ is an adjacent transposition.

When we apply a transposition to a permutation, we take the composition of the functions, which results in interchanging the two entries in the permutation which are indicated by the transposition. For example, when we apply the transposition $(4,2,3,1,5)$ to the permutation $(p_1, p_2, p_3, p_4, p_5) = (3,2,4,1,5)$, we get $(p_4, p_2, p_3, p_1, p_5) = (1,2,4,3,5)$. Given a permutation, $p = (p_1, p_2, \cdots p_n)$, there is a sequence of transpositions which bring p into ascending order. For example, here is a sequence of transpositions which bring$(3,2,4,1,5)$ into ascending order:

$$(3,2,4,1,5) \to (1,2,4,3,5) \to (1,2,3,4,5)$$

And here is a sequence of adjacent transpositions which bring$(3,2,4,1,5)$ into ascending order:

$$(3,2,4,1,5) \to (3,2,1,4,5) \to (3,1,2,4,5) \to (1,3,2,4,5) \to (1,2,3,4,5)$$

Property:

2. If there are two sequence of transpositions which bring a permutation into ascending order, then either both have an even number of transpositions or both have an odd number of transpositions.

Proof: Use induction on n. Reduce a permutation of $\mathbb{Z}_n$ to a permutation of $\mathbb{Z}_{n-1}$ roughly as follows: Preceed the sequence by 2 more transpositions, one which moves p_n out of the n^{th} position and one which moves n into the n^{th} position. Then we are starting with a permutation with n in the n^{th} position and we only need to permute the first $n-1$ positions.

A permutation is **even** (resp. **odd**) if it requires an even (resp. odd) number of transpositions to bring it into ascending order. For example,the permutation $(3,2,4,1,5)$ is even because it takes an even number of transpositions to bring it to ascending order. (See the above two sequences.) Similarly,the permutation $(2,1,4,5,3)$ is odd because it takes an odd number of transpositions to bring it to ascending order. (Try it.)

We define the **sign** or **signature** of the permutation, p, denoted by ε_p or $\varepsilon_{p_1 p_2 \cdots p_n}$, to be $+1$ if p is even and -1 if p is odd. For later purposes, we would also like to write $\varepsilon_{i_1 i_2 \cdots i_n}$ when $(i_1, i_2, \cdots, i_n)$ is not a permutation. So we define

$$\varepsilon_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is an even permutation} \\ 0 & \text{if } p \text{ is not a permutation} \end{cases}$$

For example:

$$\varepsilon_{3,2,4,1,5} = 1 \qquad \varepsilon_{2,1,4,5,3} = -1 \qquad \varepsilon_{3,2,4,2,5} = 0$$

The **inverse of a permutation**, p, denoted $\bar{p}$, is the inverse function of p. To find the inverse permutation, write a $2 \times n$ matrix with the numbers $1, 2, \cdots, n$ on the first row and the numbers $p_1, p_2, \cdots p_n$ on the second row. Rearrange the columns so the bottom numbers are in ascending order, taking the top numbers along with them. Then the top row will become $\bar{p}$:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix} \to \begin{pmatrix} \bar{p}_1 & \bar{p}_2 & \bar{p}_3 & \cdots & \bar{p}_n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

For example, to find the inverse of $p = (3, 2, 4, 1, 5)$ we write:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 4 & 2 & 1 & 3 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

So $\bar{p} = (4, 2, 1, 3, 5)$.

Property:

3. If p is an even (resp. odd) permutation, the so is $\bar{p}$.

Proof: If we perform a sequence of transpositions on columns of the $2 \times n$ matrix above to bring p into ascending order, then the same sequence of transpositions in reverse order will transfor $\bar{p}$ into ascending order.

For example:

$$\varepsilon_{4,2,1,3,5} = \varepsilon_{3,2,4,1,5} = 1$$