

Prove the Pythagorean Identity for Vectors:

$$(\vec{u} \cdot \vec{v})^2 + |\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2$$

as follows: Consider the vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$.

Compute each of the following by hand on paper. Show your work. Simplify where possible.

1. Write the algebraic definition for $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

2. Write out $(\vec{u} \cdot \vec{v})^2$ to get 6 terms.

$$\begin{aligned} (\vec{u} \cdot \vec{v})^2 &= (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\ &= u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 + 2u_1 v_1 u_2 v_2 + 2u_1 v_1 u_3 v_3 + 2u_2 v_2 u_3 v_3 \end{aligned}$$

3. Write the algebraic definition for $\vec{u} \times \vec{v}$.

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

4. Write out $|\vec{u} \times \vec{v}|^2$ to get 9 terms.

$$\begin{aligned} |\vec{u} \times \vec{v}|^2 &= (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 \\ &= (u_2^2 v_3^2 - 2u_2 v_3 u_3 v_2 + u_3^2 v_2^2) + (u_3^2 v_1^2 - 2u_3 v_1 u_1 v_3 + u_1^2 v_3^2) + (u_1^2 v_2^2 - 2u_1 v_2 u_2 v_1 + u_2^2 v_1^2) \end{aligned}$$

5. Add $(\vec{u} \cdot \vec{v})^2 + |\vec{u} \times \vec{v}|^2$ and cancel some terms.

$$\begin{aligned} (\vec{u} \cdot \vec{v})^2 + |\vec{u} \times \vec{v}|^2 &= u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 + 2u_1 v_1 u_2 v_2 + 2u_1 v_1 u_3 v_3 + 2u_2 v_2 u_3 v_3 \\ &\quad + (u_2^2 v_3^2 - 2u_2 v_3 u_3 v_2 + u_3^2 v_2^2) + (u_3^2 v_1^2 - 2u_3 v_1 u_1 v_3 + u_1^2 v_3^2) + (u_1^2 v_2^2 - 2u_1 v_2 u_2 v_1 + u_2^2 v_1^2) \\ &= u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 + u_2^2 v_3^2 + u_3^2 v_2^2 + u_3^2 v_1^2 + u_1^2 v_3^2 + u_1^2 v_2^2 + u_2^2 v_1^2 \end{aligned}$$

6. Multiply out $|\vec{u}|^2 |\vec{v}|^2$.

$$\begin{aligned} |\vec{u}|^2 |\vec{v}|^2 &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \\ &= u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2 \end{aligned}$$

7. Are the answers to (5) and (6) equal?

Yes!