MATH 221 Take Home Quiz 2 P. Yasskin

Version D Solutions

D) If your last name begins with S-Z, consider the curve $\vec{r}(t) = (t^2, 2t, \ln(t))$.
Compute each of the following. Show your work. Simplify where possible.

1. velocity
$\vec{v}(t) = \left(2t, 2, \frac{1}{t}\right)$

2. acceleration
$\vec{a}(t) = \left(2, 0, \frac{-1}{t^2}\right)$

3. jerk
$\vec{j}(t) = \left(0, 0, \frac{2}{t^3}\right)$

4. speed (HINT: The quantity in the square root is a perfect square.)
$|\vec{v}(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\left(2t + \frac{1}{t}\right)^2} = 2t + \frac{1}{t} = \frac{2t^2+1}{t}$

5. arclength between $t = 1$ and $t = 2$
$L = \int_{(1,2,0)}^{(4,4,\ln 2)} ds = \int_1^2 |\vec{v}(t)|\,dt = \int_1^2 \left(2t + \frac{1}{t}\right) dt = \left[t^2 + \ln(t)\right]_1^2$
$= (4 + \ln 2) - (1 + 0) = 3 + \ln 2$

6. unit tangent vector
$\hat{T} = \frac{\vec{v}}{|\vec{v}(t)|} = \frac{t}{2t^2+1}\vec{v} = \left(\frac{2t^2}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{1}{2t^2+1}\right)$

7. $\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t & 2 & \frac{1}{t} \\ 2 & 0 & \frac{-1}{t^2} \end{vmatrix} = \hat{\imath}\left(\frac{-2}{t^2}\right) - \hat{\jmath}\left(\frac{-2}{t} - \frac{2}{t}\right) + \hat{k}(-4) = \left(\frac{-2}{t^2}, \frac{4}{t}, -4\right)$

8. $|\vec{v} \times \vec{a}| = \sqrt{\frac{4}{t^4} + \frac{16}{t^2} + 16} = \frac{2}{t^2}\sqrt{1 + 4t^4 + 4t^4} = \frac{2(2t^2+1)}{t^2}$

9. unit binormal vector
$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{t^2}{2t^2+1}\left(\frac{-1}{t^2}, \frac{2}{t}, -2\right) = \left(\frac{-1}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{-2t^2}{2t^2+1}\right)$

10. unit normal vector

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{(2t^2+1)^2} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2t & -2t^2 \\ 2t^2 & 2t & 1 \end{vmatrix}$$

$$= \frac{1}{(2t^2+1)^2} \left[\hat{\imath}(2t+4t^3) - \hat{\jmath}(-1+4t^4) + \hat{k}(-2t-4t^3) \right] = \left(\frac{2t}{2t^2+1}, \frac{1-2t^2}{2t^2+1}, \frac{-2t}{2t^2+1} \right)$$

11. curvature

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \left(\frac{t}{2t^2+1} \right)^3 \frac{2(2t^2+1)}{t^2} = \frac{2t}{(2t^2+1)^2}$$

12. torsion

$$\tau = \frac{\vec{v} \times \vec{a} \cdot \vec{j}}{|\vec{v} \times \vec{a}|^2} = \left(\frac{t^2}{2(2t^2+1)} \right)^2 \left(\frac{-2}{t^2}, \frac{4}{t}, -4 \right) \cdot \left(0, 0, \frac{2}{t^3} \right) = \left(\frac{t^2}{2(2t^2+1)} \right)^2 \left(\frac{-8}{t^3} \right) = \frac{-2t}{(2t^2+1)^2}$$

13. tangential acceleration (compute in 2 ways)

$$a_T = \vec{a} \cdot \hat{T} = \left(2, 0, \frac{-1}{t^2} \right) \cdot \left(\frac{2t^2}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{1}{2t^2+1} \right) = \frac{4t^2}{2t^2+1} - \frac{1}{t^2(2t^2+1)} = \frac{4t^4-1}{t^2(2t^2+1)}$$

$$= \frac{2t^2-1}{t^2} = 2 - \frac{1}{t^2}$$

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(2t + \frac{1}{t} \right) = 2 - \frac{1}{t^2}$$

14. normal acceleration (compute in 2 ways)

$$a_N = \vec{a} \cdot \hat{N} = \left(2, 0, \frac{-1}{t^2} \right) \cdot \left(\frac{2t}{2t^2+1}, \frac{1-2t^2}{2t^2+1}, \frac{-2t}{2t^2+1} \right) = \frac{4t}{2t^2+1} + \frac{2}{t(2t^2+1)} = \frac{4t^2+2}{t(2t^2+1)} = \frac{2}{t}$$

$$a_N = \kappa |\vec{v}|^2 = \frac{2t}{(2t^2+1)^2} \left(2t + \frac{1}{t} \right)^2 = \frac{2t}{(2t^2+1)^2} \left(\frac{2t^2+1}{t} \right)^2 = \frac{2}{t}$$

15. mass of a wire between $t = 1$ and $t = 2$ with linear density $\rho = x$

$$M = \int_{(1,2,0)}^{(4,4,\ln 2)} \rho \, ds = \int_1^2 \rho(r(t)) |\vec{v}(t)| \, dt = \int_1^2 t^2 \left(\frac{2t^2+1}{t} \right) dt = \int_1^2 (2t^3 + t) \, dt$$

$$= \left[\frac{t^4}{2} + \frac{t^2}{2} \right]_1^2 = \left(\frac{16}{2} + \frac{4}{2} \right) - \left(\frac{1}{2} + \frac{1}{2} \right) = 9$$

16. work to move a bead along the wire from $t = 1$ to $t = 2$.
For curves B, C and D, the force is $\vec{F} = (0, y, x)$.

$$\vec{F} = (0, y, x) \qquad \vec{F}(\vec{r}(t)) = (0, 2t, t^2)$$

$$W = \int_{(1,2,0)}^{(4,4,\ln 2)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F}(r(t)) \cdot \vec{v}(t) \, dt = \int_1^2 (0, 2t, t^2) \cdot \left(2t, 2, \frac{1}{t} \right) dt$$

$$= \int_1^2 (4t + t) \, dt = \int_1^2 5t \, dt = \left[\frac{5t^2}{2} \right]_1^2 = 10 - \frac{5}{2} = \frac{15}{2}$$