MATH 221 Take Home Quiz 2 P. Yasskin

Version E Solutions

E) Anyone may consider the curve $\vec{r}(t) = (\sinh(t), \cosh(t), t)$.
Compute each of the following. Show your work. Simplify where possible.

1. velocity
$\vec{v}(t) = (\cosh(t), \sinh(t), 1)$

2. acceleration
$\vec{a}(t) = (\sinh(t), \cosh(t), 0)$

3. jerk
$\vec{j}(t) = (\cosh(t), \sinh(t), 0)$

4. speed (HINT: The quantity in the square root is a perfect square.)
$|\vec{v}(t)| = \sqrt{\cosh^2(t) + \sinh^2(t) + 1} = \sqrt{2\cosh^2(t)} = \sqrt{2}\cosh(t)$

5. arclength between $t = 1$ and $t = 2$
$$L = \int_{(\sinh(1),\cosh(1),1)}^{(\sinh(2),\cosh(2),2)} ds = \int_1^2 |\vec{v}(t)|\,dt = \int_1^2 \sqrt{2}\cosh(t)\,dt = \left[\sqrt{2}\sinh(t)\right]_1^2 = \sqrt{2}\,[\sinh(2) - \sinh(1)]$$

6. unit tangent vector
$$\hat{T} = \frac{\vec{v}}{|\vec{v}(t)|} = \frac{1}{\sqrt{2}\cosh(t)}\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{\sinh(t)}{\sqrt{2}\cosh(t)}, \frac{1}{\sqrt{2}\cosh(t)}\right)$$

7. $$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cosh(t) & \sinh(t) & 1 \\ \sinh(t) & \cosh(t) & 0 \end{vmatrix} = \hat{\imath}(-\cosh(t)) - \hat{\jmath}(-\sinh(t)) + \hat{k}(\cosh^2(t) - \sinh^2(t))$$
$$= (-\cosh(t), \sinh(t), 1)$$

8. $|\vec{v} \times \vec{a}| = \sqrt{\cosh^2(t) + \sinh^2(t) + 1} = \sqrt{2}\cosh(t)$

9. unit binormal vector
$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{1}{\sqrt{2}\cosh(t)}(-\cosh(t), \sinh(t), 1) = \left(\frac{-1}{\sqrt{2}}, \frac{\sinh(t)}{\sqrt{2}\cosh(t)}, \frac{1}{\sqrt{2}\cosh(t)}\right)$$

10. unit normal vector

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\left(\sqrt{2}\cosh(t)\right)^2} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\cosh(t) & \sinh(t) & 1 \\ \cosh(t) & \sinh(t) & 1 \end{vmatrix}$$

$$= \frac{1}{2\cosh^2(t)}\left[\hat{\imath}(\sinh(t) - \sinh(t)) - \hat{\jmath}(-\cosh(t) - \cosh(t)) + \hat{k}(-\cosh(t)\sinh(t) - \sinh(t)\cosh(t))\right]$$

$$= \frac{1}{2\cosh^2(t)}(0, 2\cosh(t), -2\sinh(t)\cosh(t)) = \left(0, \frac{1}{\cosh(t)}, -\frac{\sinh(t)}{\cosh(t)}\right)$$

11. curvature

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{2}\cosh(t)}{\left(\sqrt{2}\cosh(t)\right)^3} = \frac{1}{2\cosh^2(t)}$$

12. torsion

$$\tau = \frac{\vec{v} \times \vec{a} \cdot \vec{j}}{|\vec{v} \times \vec{a}|^2} = \left(\frac{1}{\sqrt{2}\cosh(t)}\right)^2 (-\cosh(t), \sinh(t), 1) \cdot (\cosh(t), \sinh(t), 0)$$

$$= \frac{1}{2\cosh^2(t)}(-\cosh^2(t) + \sinh^2(t)) = \frac{-1}{2\cosh^2(t)}$$

13. tangential acceleration (compute in 2 ways)

$$a_T = \vec{a} \cdot \hat{T} = (\sinh(t), \cosh(t), 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{\sinh(t)}{\sqrt{2}\cosh(t)}, \frac{1}{\sqrt{2}\cosh(t)}\right) = \frac{\sinh(t)}{\sqrt{2}} + \frac{\sinh(t)}{\sqrt{2}}$$

$$= \sqrt{2}\sinh(t)$$

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(\sqrt{2}\cosh(t)\right) = \sqrt{2}\sinh(t)$$

14. normal acceleration (compute in 2 ways)

$$a_N = \vec{a} \cdot \hat{N} = (\sinh(t), \cosh(t), 0) \cdot \left(0, \frac{1}{\cosh(t)}, -\frac{\sinh(t)}{\cosh(t)}\right) = 1$$

$$a_N = \kappa|\vec{v}|^2 = \frac{1}{2\cosh^2(t)}\left(\sqrt{2}\cosh(t)\right)^2 = 1$$

15. mass of a wire between $t = 1$ and $t = 2$ with linear density $\rho = x$

$$M = \int_{(\sinh(1),\cosh(1),1)}^{(\sinh(2),\cosh(2),2)} \rho\, ds = \int_1^2 \rho(r(t))|\vec{v}(t)|\, dt = \int_1^2 \sinh(x)\left(\sqrt{2}\cosh(t)\right) dt = \left[\sqrt{2}\,\frac{\sinh^2(t)}{2}\right]_1^2$$

$$= \frac{1}{\sqrt{2}}(\sinh^2(2) - \sinh^2(1))$$

16. work to move a bead along the wire from $t = 1$ to $t = 2$.
For curves A and E, the force is $\vec{F} = (-y, x, 0)$.

$$\vec{F} = (-y, x, 0) \qquad \vec{F}(\vec{r}(t)) = (-\cosh(t), \sinh(t), 0)$$

$$W = \int_{(\sinh(1),\cosh(1),1)}^{(\sinh(2),\cosh(2),2)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F}(r(t)) \cdot \vec{v}(t)\, dt = \int_1^2 (-\cosh(t), \sinh(t), 0) \cdot (\cosh(t), \sinh(t), 1)\, dt$$

$$= \int_1^2 (-\cosh^2(t) + \sinh^2(t))\, dt = \int_1^2 (-1)\, dt = [-t]_1^2 = -2 + 1 = -1$$