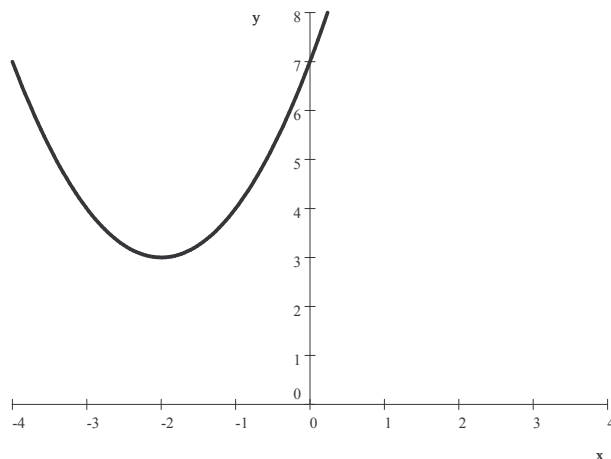


Multiple Choice: (4 points each. No part credit. No calculator.)

1. The plot at the right is which function?



- a. $(x + 2)^2 + 3$ CORRECT
- b. $(x + 2)^2 - 3$
- c. $(x - 2)^2 + 3$
- d. $(x - 3)^2 + 2$
- e. $(x + 3)^2 + 2$

The function x^2 is shifted left by 2 and up by 3. So at $x = -2$ the value must be 3.

2. Find the angle between the vectors $\vec{u} = (\sqrt{3}, 1)$ and $\vec{v} = (\sqrt{3} - 1, \sqrt{3} + 1)$.

- a. 30°
- b. 45° CORRECT
- c. 60°
- d. 120°
- e. 135°

$$\vec{u} \cdot \vec{v} = \sqrt{3}(\sqrt{3} - 1) + 1(\sqrt{3} + 1) = 3 - \sqrt{3} + \sqrt{3} + 1 = 4$$

$$|\vec{u}| = \sqrt{3+1} = 2$$

$$|\vec{v}| = \sqrt{(\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2} = \sqrt{(3 - 2\sqrt{3} + 1) + (3 + 2\sqrt{3} + 1)} = \sqrt{8} = 2\sqrt{2}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{4}{2 \cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \theta = 45^\circ$$

3. Compute $\lim_{x \rightarrow 3} \frac{x-3}{(x^2-9)}$

a. $\frac{1}{6}$ CORRECT

b. $\frac{1}{3}$

c. 0

d. $-\frac{1}{3}$

e. $-\frac{1}{6}$

$$\lim_{x \rightarrow 3} \frac{x-3}{(x^2-9)} = \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{1}{(x+3)} = \frac{1}{6}$$

4. For the function $f(x) = \begin{cases} 5+x & \text{if } x \leq 3 \\ x^2-2 & \text{if } x > 3 \end{cases}$ which of the following is FALSE?

a. $\lim_{x \rightarrow 3^-} f(x) = 8$

b. $\lim_{x \rightarrow 3^+} f(x) = 7$

c. $f(3) = 8$

d. f is continuous from the left at $x = 3$

e. f is continuous from the right at $x = 3$ CORRECT

f is NOT continuous from the right at $x = 3$ because $f(3) = 8$ but $\lim_{x \rightarrow 3^+} f(x) = 7$

5. Compute $\lim_{x \rightarrow 9^-} \frac{x+9}{x^2(x-9)}$

a. $-\infty$ CORRECT

b. $-\frac{2}{9}$

c. 0

d. $\frac{2}{9}$

e. ∞

$$\lim_{x \rightarrow 9^-} \frac{x+9}{x^2(x-9)} = \frac{(18^-)}{(9^-)^2(0^-)} = -\infty$$

6. If $f(x)$ satisfies $7x \leq f(x) \leq x^2 + 6$ for all $x \neq 1$ and $f(1) = 5$, then $\lim_{x \rightarrow 1} f(x) =$
- undefined
 - 1
 - 5
 - 6
 - 7 CORRECT

Since $\lim_{x \rightarrow 1} 7x = 7$ and $\lim_{x \rightarrow 1} (x^2 + 6) = 7$, the Squeeze Theorem says $\lim_{x \rightarrow 1} f(x) = 7$ also.

7. Compute $\lim_{x \rightarrow e} \frac{\ln x - 1}{x - e}$
- $-e$
 - $-\frac{1}{e}$
 - 0
 - $\frac{1}{e}$ CORRECT
 - $\frac{1}{2e}$

$$\lim_{x \rightarrow e} \frac{\ln x - 1}{x - e} \stackrel{L'H}{=} \lim_{x \rightarrow e} \frac{\frac{1}{x}}{1} = \frac{1}{e}$$

8. If $f(x) = \frac{\cos x}{1 + \sin x}$ then $f'(x) =$
- $\frac{\sin x + 1}{\cos^2 x}$
 - $\frac{-\sin x + 1}{(1 + \sin x)^2}$
 - $\frac{-\sin x - 1}{(1 + \sin x)^2}$ CORRECT
 - $\frac{\sin x + 1}{(1 + \sin x)^2}$
 - $\frac{\sin x - 1}{(1 + \sin x)^2}$

$$f'(x) = \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} = \frac{-\sin x - 1}{(1 + \sin x)^2}$$

9. If $f(x) = x^{\tan x}$ then $f'(x) =$

- a. $(\tan x)x^{\tan x - 1}$
- b. $x^{\tan x} \left[\frac{\tan x}{x} + \ln x \sec^2 x \right]$ CORRECT
- c. $(\ln x)x^{\tan x}$
- d. $(\ln x)x^{\tan x} \sec^2 x$
- e. $x^{\tan x} \left[\frac{\tan x}{x} + \ln(\sec^2 x) \right]$

$$f(x) = x^{\tan x} = (e^{\ln x})^{\tan x} = e^{\ln x \tan x}$$

$$\text{So } f'(x) = e^{\ln x \tan x} \left[\frac{1}{x} \tan x + \ln x \sec^2 x \right] = x^{\tan x} \left[\frac{1}{x} \tan x + \ln x \sec^2 x \right]$$

10. Find the critical numbers of the function $f(x) = |2x - x^2|$.

- a. -2
- b. 0, 1
- c. 0, 2
- d. 0, 1, 2 CORRECT
- e. -2, 0, 2

If $2x - x^2 > 0$ then $f(x) = 2x - x^2$ and $f'(x) = 2 - 2x$.

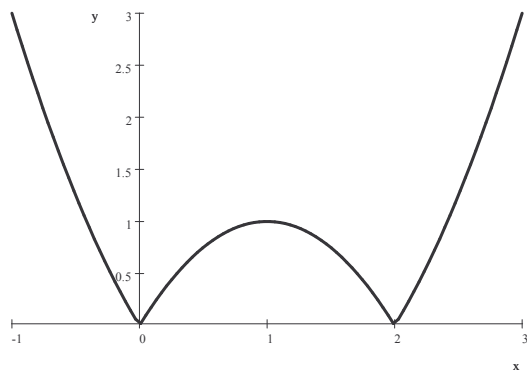
So $x = 1$ is critical. Check: $2x - x^2 = 2(1) - 1^2 = 1 > 0$.

If $2x - x^2 < 0$ then $f(x) = -(2x - x^2)$ and $f'(x) = -2 + 2x$.

So $x = 1$ is critical also, but: $2x - x^2 = 2(1) - 1^2 = 1 \not< 0$. So this case is irrelevant.

If $2x - x^2 = 0$ then $f(x)$ is undefined. $x(2 - x) = 0$.

So $x = 0$ and $x = 2$ are also critical.



11. If a car starts from rest at $x(0) = 0$ mi, and accelerates (and decelerates) at $a(t) = \sin(t)$ mi/hr², how far does it travel in 2π hours?

- a. 1 mi
- b. 2 mi
- c. $\frac{\pi}{2}$ mi
- d. π mi
- e. 2π mi CORRECT

$$v(t) = -\cos(t) + C \quad v(0) = -1 + C = 0 \quad C = 1 \quad v(t) = -\cos(t) + 1$$

$$x(t) = -\sin(t) + t + D \quad x(0) = D = 0 \quad x(t) = -\sin(t) + t \quad x(2\pi) = 2\pi$$

12. Compute $\int_4^9 (x^{1/2} + x^{-1/2}) dx$

- a. $-\frac{17}{432}$
- b. $\frac{17}{432}$
- c. $\frac{44}{3}$ CORRECT
- d. 24
- e. $\frac{4}{27}$

$$\int_4^9 (x^{1/2} + x^{-1/2}) dx = \left[\frac{2x^{3/2}}{3} + 2x^{1/2} \right]_4^9 = \left[\frac{2 \cdot 27}{3} + 2 \cdot 3 \right] - \left[\frac{2 \cdot 8}{3} + 2 \cdot 2 \right] = \frac{44}{3}$$

13. Compute $\int_0^{1/2} \frac{e^{2x}}{4 + e^{2x}} dx$

- a. $\ln \sqrt{5(4 + e)}$
- b. $\ln \sqrt{\frac{5}{4 + e}}$
- c. $\ln \sqrt{\frac{4 + e}{5}}$ CORRECT
- d. $\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \ln 0$
- e. $\frac{1}{2} \ln \frac{1}{2}$

$$u = 4 + e^{2x} \quad du = 2e^{2x} dx \quad \frac{1}{2} du = e^{2x} dx$$

$$\int_0^{1/2} \frac{e^{2x}}{4 + e^{2x}} dx = \frac{1}{2} \int_5^{4+e} \frac{1}{u} du = \left[\frac{1}{2} \ln|u| \right]_5^{4+e} = \frac{1}{2} \ln(4 + e) - \frac{1}{2} \ln(5) = \ln \sqrt{\frac{4 + e}{5}}$$

Work Out: (10 points each. Part credit possible. Calculators allowed. Show all work.)

14. Find the equation of the line tangent to $y = \frac{4e^x}{x}$ at $x = 2$.

$$f(x) = \frac{4e^x}{x} \quad f(2) = 2e^2 \quad f'(x) = \frac{x4e^x - 4e^x}{x^2} \quad f'(2) = 2e^2 - e^2 = e^2$$

$$y = f(2) + f'(2)(x - 2) = 2e^2 + e^2(x - 2) = e^2x$$

15. If you start with 400 kg of radioactive element X which has a half life of 20 years, how much X will there be after 30 years?

$$y(t) = 400e^{-kt} \quad 200 = 400e^{-k20} \quad e^{-k20} = \frac{1}{2} \quad k = \frac{-1}{20} \ln \frac{1}{2} = \frac{\ln 2}{20}$$

$$y(t) = 400e^{-t(\ln 2)/20} \quad y(30) = 400e^{-30(\ln 2)/20} = 400 \cdot 2^{-3/2} = \frac{400}{2\sqrt{2}} = 100\sqrt{2}$$

$$\text{OR} \quad y(t) = 400\left(\frac{1}{2}\right)^{t/20} \quad y(30) = 400\left(\frac{1}{2}\right)^{30/20} = \frac{400}{2\sqrt{2}} = 100\sqrt{2}$$

16. The position and velocity of a mass hanging from a spring are related by $4x^2 + v^2 = 100$. At time $t = 7$, the position is $x(7) = 4$ and **increasing**.

- a. (3 points) Find the velocity at $t = 7$, i.e. find $v(7) = \frac{dx}{dt}(7)$.

$$v = \sqrt{100 - 4x^2} \quad \text{We use the positive square root because } x \text{ is increasing.}$$

$$v(7) = \sqrt{100 - 4(4)^2} = \sqrt{100 - 64} = 6$$

- b. (5 points) Find the acceleration at $t = 7$, i.e. find $a(7) = \frac{dv}{dt}(7)$.

Apply $\frac{d}{dt}$ to both sides of $4x^2 + v^2 = 100$, evaluate at $t = 7$ and solve for the acceleration:

$$8x \frac{dx}{dt} + 2v \frac{dv}{dt} = 0 \quad \Rightarrow \quad 8(4)(6) + 2(6) \frac{dv}{dt} = 0$$

$$\Rightarrow \quad a(7) = \frac{dv}{dt}(7) = -\frac{8(4)(6)}{2(6)} = -16$$

- c. (2 points) Is the velocity increasing or decreasing? Why?

Since a is negative, the velocity decreasing.

17. Find the area of the largest rectangle that can be inscribed in the triangle with vertices $(0,0)$, $(0,3)$ and $(4,0)$, if two edges of the rectangle are along the axes. Explain why your critical point is a local maximum.

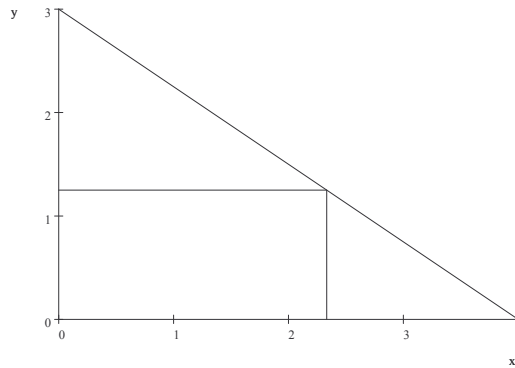
The hypotenuse is $y = 3 - \frac{3}{4}x$

We must maximize the area:

$$A = xy = x\left(3 - \frac{3}{4}x\right) = 3x - \frac{3}{4}x^2$$

$$A' = 3 - \frac{3}{2}x = 0 \Rightarrow x = 2$$

$$A = 3(2) - \frac{3}{4}(2)^2 = 3$$



$A'' = -\frac{3}{2} < 0$ So A is everywhere concave down and $x = 2$ is a local and global maximum.

OR $A = 3x - \frac{3}{4}x^2$ is a parabola opening down and so $x = 2$ is a global maximum.

18. Use the Method of Riemann Sums with equal intervals and Right Endpoints to compute the integral $\int_1^4 x(x-1) dx$.

Use the F.T.C. only to check your answer.

Hints: $\sum_{i=1}^n 1 = n$ $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$\Delta x = \frac{4-1}{n} = \frac{3}{n} \quad x_i = 1 + i\Delta x = 1 + \frac{3i}{n}$$

$$f(x) = x(x-1) \quad f(x_i) = \left(1 + \frac{3i}{n}\right)\frac{3i}{n} = \frac{3i}{n} + \frac{9i^2}{n^2}$$

$$\begin{aligned} \int_1^4 x(x-1) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{3i}{n} + \frac{9i^2}{n^2}\right) \frac{3}{n} = \lim_{n \rightarrow \infty} \left(\frac{9}{n^2} \sum_{i=1}^n i + \frac{27}{n^3} \sum_{i=1}^n i^2\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{9}{n^2} \frac{n(n+1)}{2} + \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6}\right) = \lim_{n \rightarrow \infty} \left(\frac{9}{2} \frac{(n+1)}{n} + \frac{9}{2} \frac{(n+1)(2n+1)}{n^2}\right) \\ &= \frac{9}{2} + 9 = \frac{27}{2} \end{aligned}$$

Check: $\int_1^4 x(x-1) dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^4 = \left[\frac{64}{3} - \frac{16}{2}\right] - \left[\frac{1}{3} - \frac{1}{2}\right] = \left[21 - \frac{15}{2}\right] = \frac{27}{2}$.