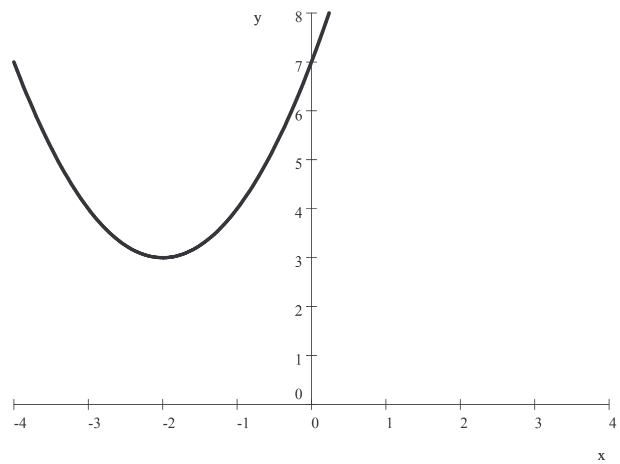
Name____________________ ID______________ SEC________

MATH 151 Final Exam Version A Fall 2004

Sections 501-503, 515-517 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit. No calculator.)

1. The plot at the right is which function?



a. $(x+2)^2+3$ CORRECT

b. $(x+2)^2-3$

c. $(x-2)^2+3$

d. $(x-3)^2+2$

e. $(x+3)^2+2$

The function x^2 is shifted left by 2 and up by 3. So at $x=-2$ the value must be 3.

2. Find the angle between the vectors $\vec{u} = \left(\sqrt{3}, 1\right)$ and $\vec{v} = \left(\sqrt{3}-1, \sqrt{3}+1\right)$.

a. $30°$

b. $45°$ CORRECT

c. $60°$

d. $120°$

e. $135°$

$$\vec{u}\bullet\vec{v} = \sqrt{3}\left(\sqrt{3}-1\right)+1\left(\sqrt{3}+1\right) = 3-\sqrt{3}+\sqrt{3}+1 = 4$$

$$|\vec{u}| = \sqrt{3+1} = 2$$

$$|\vec{v}| = \sqrt{\left(\sqrt{3}-1\right)^2+\left(\sqrt{3}+1\right)^2} = \sqrt{\left(3+2\sqrt{3}+1\right)+\left(3-2\sqrt{3}+1\right)} = \sqrt{8} = 2\sqrt{2}$$

$$\cos\theta = \frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|} = \frac{4}{2\bullet 2\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \theta = 45°$$

3. Compute $\lim\limits_{x \to 3} \dfrac{x-3}{(x^2-9)}$

a. $\frac{1}{6}$ CORRECT

b. $\frac{1}{3}$

c. 0

d. $-\frac{1}{3}$

e. $-\frac{1}{6}$

$$\lim_{x \to 3} \frac{x-3}{(x^2-9)} = \lim_{x \to 3} \frac{x-3}{(x-3)(x+3)} = \lim_{x \to 3} \frac{1}{(x+3)} = \frac{1}{6}$$

4. For the function $f(x) = \begin{cases} 5+x & \text{if } x \le 3 \\ x^2-2 & \text{if } x > 3 \end{cases}$ which of the following is FALSE?

a. $\lim\limits_{x \to 3^-} f(x) = 8$

b. $\lim\limits_{x \to 3^+} f(x) = 7$

c. $f(3) = 8$

d. f is continuous from the left at $x = 3$

e. f is continuous from the right at $x = 3$ CORRECT

f is NOT continuous from the right at $x = 3$ because $f(3) = 8$ but $\lim\limits_{x \to 3^+} f(x) = 7$

5. Compute $\lim\limits_{x \to 9^-} \dfrac{x+9}{x^2(x-9)}$

a. $-\infty$ CORRECT

b. $-\frac{2}{9}$

c. 0

d. $\frac{2}{9}$

e. ∞

$$\lim_{x \to 9^-} \frac{x+9}{x^2(x-9)} = \frac{(18^-)}{(9^-)^2(0^-)} = -\infty$$

6. If $f(x)$ satisfies $7x \le f(x) \le x^2 + 6$ for all $x \ne 1$ and $f(1) = 5$, then $\lim_{x\to 1} f(x) =$

a. undefined

b. 1

c. 5

d. 6

e. 7 CORRECT

Since $\lim_{x\to 1} 7x = 7$ and $\lim_{x\to 1}(x^2 + 6) = 7$, the Squeeze Theorem says $\lim_{x\to 1} f(x) = 7$ also.

7. Compute $\lim_{x\to e} \dfrac{\ln x - 1}{x - e}$

a. $-e$

b. $-\dfrac{1}{e}$

c. 0

d. $\dfrac{1}{e}$ CORRECT

e. $\dfrac{1}{2e}$

$$\lim_{x\to e} \frac{\ln x - 1}{x - e} \overset{l'H}{=} \lim_{x\to e} \frac{\frac{1}{x}}{1} = \frac{1}{e}$$

8. If $f(x) = \dfrac{\cos x}{1 + \sin x}$ then $f'(x) =$

a. $\dfrac{\sin x + 1}{\cos^2 x}$

b. $\dfrac{-\sin x + 1}{(1 + \sin x)^2}$

c. $\dfrac{-\sin x - 1}{(1 + \sin x)^2}$ CORRECT

d. $\dfrac{\sin x + 1}{(1 + \sin x)^2}$

e. $\dfrac{\sin x - 1}{(1 + \sin x)^2}$

$$f'(x) = \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} = \frac{-\sin x - 1}{(1 + \sin x)^2}$$

9. If $f(x) = x^{\tan x}$ then $f'(x) =$

a. $(\tan x)x^{\tan x-1}$

b. $x^{\tan x}\left[\frac{\tan x}{x} + \ln x \sec^2 x\right]$ CORRECT

c. $(\ln x)x^{\tan x}$

d. $(\ln x)x^{\tan x}\sec^2 x$

e. $x^{\tan x}\left[\frac{\tan x}{x} + \ln(\sec^2 x)\right]$

$f(x) = x^{\tan x} = (e^{\ln x})^{\tan x} = e^{\ln x \tan x}$

So $f'(x) = e^{\ln x \tan x}\left[\frac{1}{x}\tan x + \ln x \sec^2 x\right] = x^{\tan x}\left[\frac{1}{x}\tan x + \ln x \sec^2 x\right]$

10. Find the critical numbers of the function $f(x) = |2x - x^2|$.

a. -2

b. $0, 1$

c. $0, 2$

d. $0, 1, 2$ CORRECT

e. $-2, 0, 2$

If $2x - x^2 > 0$ then $f(x) = 2x - x^2$ and $f'(x) = 2 - 2x$.

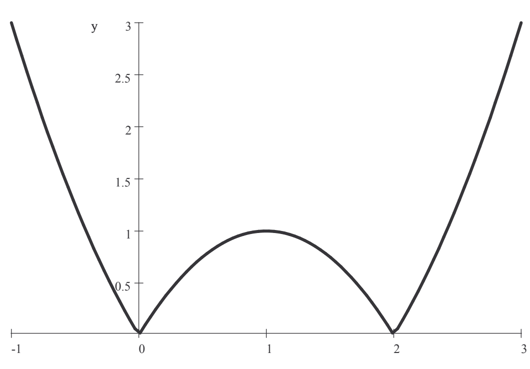
So $x = 1$ is critical. Check: $2x - x^2 = 2(1) - 1^2 = 1 > 0$.

If $2x - x^2 < 0$ then $f(x) = -(2x - x^2)$ and $f'(x) = -2 + 2x$.

So $x = 1$ is critical also,but: $2x - x^2 = 2(1) - 1^2 = 1 \not< 0$. So this case is irrelevant.

If $2x - x^2 = 0$ then $f(x)$ is undefined. $x(2 - x) = 0$.

So $x = 0$ and $x = 2$ are also critical.



11. If a car starts from rest at $x(0) = 0$ mi, and accelerates (and decelerates) at $a(t) = \sin(t)$ mi/hr^2, how far does it travel in 2π hours?

a. 1 mi

b. 2 mi

c. $\frac{\pi}{2}$ mi

d. π mi

e. 2π mi CORRECT

$v(t) = -\cos(t) + C \qquad v(0) = -1 + C = 0 \qquad C = 1 \qquad v(t) = -\cos(t) + 1$

$x(t) = -\sin(t) + t + D \qquad x(0) = D = 0 \qquad x(t) = -\sin(t) + t \qquad x(2\pi) = 2\pi$

12. Compute $\int_4^9 (x^{1/2} + x^{-1/2})\,dx$

a. $-\frac{17}{432}$

b. $\frac{17}{432}$

c. $\frac{44}{3}$ CORRECT

d. 24

e. $\frac{4}{27}$

$$\int_4^9 (x^{1/2} + x^{-1/2})\,dx = \left[\frac{2x^{3/2}}{3} + 2x^{1/2}\right]_4^9 = \left[\frac{2 \cdot 27}{3} + 2 \cdot 3\right] - \left[\frac{2 \cdot 8}{3} + 2 \cdot 2\right] = \frac{44}{3}$$

13. Compute $\int_0^{1/2} \frac{e^{2x}}{4 + e^{2x}}\,dx$

a. $\ln\sqrt{5(4+e)}$

b. $\ln\sqrt{\frac{5}{4+e}}$

c. $\ln\sqrt{\frac{4+e}{5}}$ CORRECT

d. $\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln 0$

e. $\frac{1}{2}\ln\frac{1}{2}$

$u = 4 + e^{2x} \qquad du = 2e^{2x}\,dx \qquad \frac{1}{2}du = e^{2x}\,dx$

$$\int_0^{1/2} \frac{e^{2x}}{4 + e^{2x}}\,dx = \frac{1}{2}\int_5^{4+e} \frac{1}{u}\,du = \left[\frac{1}{2}\ln|u|\right]_5^{4+e} = \frac{1}{2}\ln(4+e) - \frac{1}{2}\ln(5) = \ln\sqrt{\frac{4+e}{5}}$$

Work Out: (10 points each. Part credit possible. Calculators allowed. Show all work.)

14. Find the equation of the line tangent to $y = \frac{4e^x}{x}$ at $x = 2$.

$f(x) = \frac{4e^x}{x}$ $\quad f(2) = 2e^2$ $\quad f'(x) = \frac{x4e^x - 4e^x}{x^2}$ $\quad f'(2) = 2e^2 - e^2 = e^2$

$y = f(2) + f'(2)(x-2) = 2e^2 + e^2(x-2) = e^2 x$

15. If you start with 400 kg of radioactive element X which has a half life of 20 years, how much X will there be after 30 years?

$y(t) = 400e^{-kt}$ $\quad 200 = 400e^{-k20}$ $\quad e^{-k20} = \frac{1}{2}$ $\quad k = \frac{-1}{20}\ln\frac{1}{2} = \frac{\ln 2}{20}$

$y(t) = 400e^{-t(\ln 2)/20}$ $\quad y(30) = 400e^{-30(\ln 2)/20} = 400 \cdot 2^{-3/2} = \frac{400}{2\sqrt{2}} = 100\sqrt{2}$

OR $\quad y(t) = 400\left(\frac{1}{2}\right)^{t/20}$ $\quad y(30) = 400\left(\frac{1}{2}\right)^{30/20} = \frac{400}{2\sqrt{2}} = 100\sqrt{2}$

16. The position and velocity of a mass hanging from a spring are related by $4x^2 + v^2 = 100$. At time $t = 7$, the position is $x(7) = 4$ and **increasing**.

a. (3 points) Find the velocity at $t = 7$, i.e. find $v(7) = \frac{dx}{dt}(7)$.

$v = \sqrt{100 - 4x^2}$ We use the positive square root because x is increasing.
$v(7) = \sqrt{100 - 4(4)^2} = \sqrt{100-64} = 6$

b. (5 points) Find the acceleration at $t = 7$, i.e. find $a(7) = \frac{dv}{dt}(7)$.

Apply $\frac{d}{dt}$ to both sides of $4x^2 + v^2 = 100$, evaluate at $t = 7$ and solve for the acceleration:

$8x\frac{dx}{dt} + 2v\frac{dv}{dt} = 0 \quad \Rightarrow \quad 8(4)(6) + 2(6)\frac{dv}{dt} = 0$

$\Rightarrow \quad a(7) = \frac{dv}{dt}(7) = -\frac{8(4)(6)}{2(6)} = -16$

c. (2 points) Is the velocity increasing or decreasing? Why?

Since a is negative, the velocity decreasing.

17. Find the area of the largest rectangle that can be inscribed in the triangle with vertices $(0,0)$, $(0,3)$ and $(4,0)$, if two edges of the rectangle are along the axes. Explain why your critical point is a local maximum.

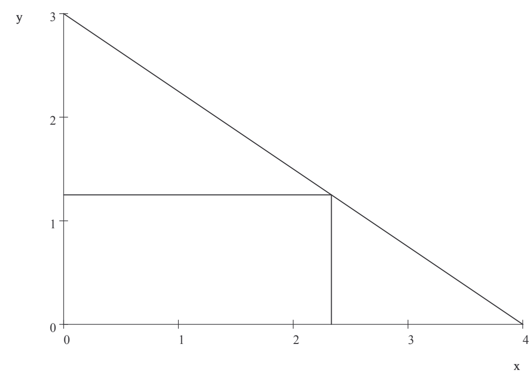
The hypotenuse is $y = 3 - \frac{3}{4}x$

We must maximize the area:

$$A = xy = x\left(3 - \frac{3}{4}\right) = 3x - \frac{3}{4}x^2$$

$$A' = 3 - \frac{3}{2}x = 0 \quad \Rightarrow \quad x = 2$$

$$A = 3(2) - \frac{3}{4}(2)^2 = 3$$



$A'' = -\frac{3}{2} < 0$ So A is everywhere concave down and $x = 2$ is a local and global maximum.

OR $A = 3x - \frac{3}{4}x^2$ is a parabola opening down and so $x = 2$ is a global maximum.

18. Use the Method of Riemann Sums with equal intervals and Right Endpoints to compute the integral $\int_1^4 x(x-1)\,dx$.

Use the F.T.C. only to check your answer.

Hints: $\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$\Delta x = \frac{4-1}{n} = \frac{3}{n} \qquad x_i = 1 + i\Delta x = 1 + \frac{3i}{n}$$

$$f(x) = x(x-1) \qquad f(x_i) = \left(1 + \frac{3i}{n}\right)\frac{3i}{n} = \frac{3i}{n} + \frac{9i^2}{n^2}$$

$$\int_1^4 x(x-1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3i}{n} + \frac{9i^2}{n^2}\right)\frac{3}{n} = \lim_{n\to\infty}\left(\frac{9}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{n^2}\frac{n(n+1)}{2} + \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}\right) = \lim_{n\to\infty}\left(\frac{9}{2}\frac{(n+1)}{n} + \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right)$$

$$= \frac{9}{2} + 9 = \frac{27}{2}$$

Check: $\int_1^4 x(x-1)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^4 = \left[\frac{64}{3} - \frac{16}{2}\right] - \left[\frac{1}{3} - \frac{1}{2}\right] = \left[21 - \frac{15}{2}\right] = \frac{27}{2}.$