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MATH 151 Final Exam Version B Fall 2004

Sections 501-503, 515-517 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit. No calculator.)

**1.** Find $x$ so that $(2,3)+x(2,-1)=(4,1)$

**a.** $x=1$ or $2$

**b.** $x=1$ only

**c.** $x=2$ only

**d.** $x=-1$ only

**e.** No solutions CORRECT

We need to solve $2+2x=4$ and $3-x=1$. Now, $x=1$ satisfies the first and $x=2$ satisfies the second, but nothing satisfies both.

**2.** Find an equation of the line through the point $P=(1,-2,3)$ which is parallel to the vector $\overrightarrow{AB}$, where $A=(4,2,1)$ and $B=(1,-3,2)$.

**a.** $(x,y,z)=(4-3t,2-5t,1+t)$

**b.** $(x,y,z)=(4+t,2-2t,1+3t)$

**c.** $(x,y,z)=(1-3t,-2-5t,3+t)$ CORRECT

**d.** $(x,y,z)=(1+4t,-2+2t,3+t)$

**e.** $(x,y,z)=(1+t,-2-3t,3+2t)$

$\overrightarrow{AB}=B-A=(1,-3,2)-(4,2,1)=(-3,-5,1)$

$X=P+t\overrightarrow{AB}=(1,-2,3)+t(-3,-5,1) \qquad (x,y,z)=(1-3t,-2-5t,3+t)$

**3.** Compute $\displaystyle\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}$

**a.** $\frac{1}{4}$ CORRECT

**b.** $\frac{1}{2}$

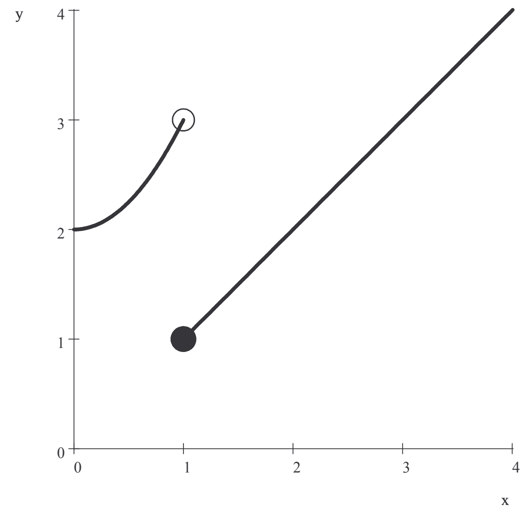
**c.** $0$

**d.** $-\frac{1}{2}$

**e.** $-\frac{1}{4}$

$$\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to4}\frac{x-4}{x-4}\cdot\frac{1}{\sqrt{x}+2}=\lim_{x\to4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

**4.** The graph of $y = f(x)$ is shown at the right.
Which of the following is FALSE?




**a.** $\lim_{x \to 1^-} f(x) = 3$

**b.** $\lim_{x \to 1^+} f(x) = 1$

**c.** $f(1) = 1$

**d.** $f$ is continuous from the left at $x = 1$ CORRECT

**e.** $f$ is continuous from the right at $x = 1$

$f$ is NOT continuous from the left at $x = 1$ because $f(1) = 1$ but $\lim_{x \to 1^-} f(x) = 3$

**5.** Which of the following is TRUE?

**a.** $\lim_{x \to -11} |x + 11| = 11$ False because the limit $= 0$.

**b.** $\lim_{x \to 11^-} \frac{|x - 11|}{x - 11} = -1$ CORRECT True because the limit is qualitatively "$\frac{|0^-|}{0^-}$".

**c.** $\lim_{x \to -11^-} \frac{|x + 11|}{x + 11} = 11$ False because the limit is qualitatively "$\frac{|0^-|}{0^-} = -1$".

**d.** $\lim_{x \to 11} \frac{|x - 11|}{x - 11} = 1$ False because the the two one sided limits are $1$ and $-1$.

**e.** $\lim_{x \to -11^-} \frac{|x + 11|}{x + 11} = 0$ False because the limit is qualitatively "$\frac{|0^-|}{0^-} = -1$".

**6.** Compute $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h}$.

HINT: This limit is $f'(a)$ for what $f$ and what $a$?

**a.** $4$

**b.** $2$ CORRECT

**c.** $\frac{4}{3}$

**d.** $1$

**e.** $\frac{1}{2}$

If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$ and

$$\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h} = f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2.$$

**7.** Compute $\lim_{x \to 1} \frac{e^x - e}{x - 1}$

**a.** $e$ CORRECT

**b.** $\frac{1}{e}$

**c.** $0$

**d.** $2e$

**e.** $\frac{1}{2e}$

$$\lim_{x \to 1} \frac{e^x - e}{x - 1} \overset{l'H}{=} \lim_{x \to 1} \frac{e^x}{1} = e$$

**8.** If $f(x) = \frac{7x - 1}{9x + 2}$ then $f'(x) =$

**a.** $\frac{-5}{(9x+2)^2}$

**b.** $\frac{-5}{(7x-1)^2}$

**c.** $\frac{5}{(7x-1)^2}$

**d.** $\frac{-23}{(9x+2)^2}$

**e.** $\frac{23}{(9x+2)^2}$ CORRECT

$$f'(x) = \frac{(9x+2)7 - (7x-1)9}{(9x+2)^2} = \frac{2 \cdot 7 + 9}{(9x+2)^2} = \frac{23}{(9x+2)^2}$$

**9.** If $f(x) = (\sin x)^{3x}$ then $f'(x) =$

**a.** $(\sin x)^{3x}[3 \ln(\sin x) + 3x^2 \tan x]$

**b.** $(\sin x)^{3x}[3 \ln x \cos x + 3x \tan x]$

**c.** $(\sin x)^{3x}[3 \ln(\cos x) + 3x \cot x]$

**d.** $(\sin x)^{3x}[3 \ln(\sin x) + 3x \cot x]$ CORRECT

**e.** $(\sin x)^{3x}[3 \ln x \sin x + 3x \cot x]$

$$f(x) = (\sin x)^{3x} = (e^{\ln \sin x})^{3x} = e^{3x \ln \sin x}$$

So $f'(x) = e^{3x \ln \sin x}\left[3 \ln \sin x + \frac{3x}{\sin x} \cos x\right] = (\sin x)^{3x}[3 \ln \sin x + 3x \cot x]$

**10.** Find the critical numbers of the function $f(x) = x^{1/4}(x-3)^2$.

**a.** $3, \frac{1}{3}$

**b.** $3, \frac{1}{3}, 1$

**c.** $3, \frac{1}{3}, 0$ CORRECT

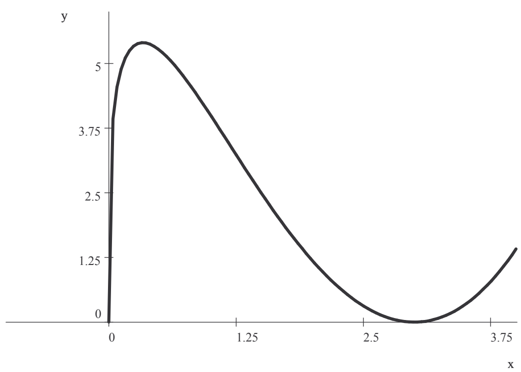
**d.** $3, 0$

**e.** $3, 0, -3$

$f'(x) = x^{1/4}2(x-3) + (x-3)^2 \frac{1}{4}x^{-3/4}$

$= \dfrac{x8(x-3) + (x-3)^2}{4x^{3/4}} = \dfrac{(x-3)(x8 + x - 3)}{4x^{3/4}} = \dfrac{(x-3)(9x-3)}{4x^{3/4}}$

Critical numbers are $x = 3, \frac{1}{3}, 0.$




**11.** A ball is dropped (initial velocity $v(0) = 0$) from the top of a tall building. Due to air resistance, its acceleration is only $a(t) = 6 + 4e^{-t}$ m/sec$^2$. How far does it fall in $t = 1$ sec?

**a.** $3 - 4e^{-1}$ m

**b.** $3 + 4e^{-1}$ m CORRECT

**c.** $10 - 4e^{-1}$ m

**d.** $10 + 4e^{-1}$ m

**e.** $6 + 4e^{-1}$ m

$v(t) = 6t - 4e^{-t} + C \qquad v(0) = -4 + C = 0 \qquad C = 4 \qquad v(t) = 6t - 4e^{-t} + 4$

$x(t) = 3t^2 + 4e^{-t} + 4t + D \qquad x(0) = 4 + D = 0 \qquad D = -4$

$x(t) = 3t^2 + 4e^{-t} + 4t - 4 \qquad x(1) = 3 + 4e^{-1} + 4 - 4 = 3 + 4e^{-1}$

**12.** Compute $\int_0^{\pi/2} \sin(2x)\,dx$

**a.** $-\pi$

**b.** $-1$

**c.** $0$

**d.** $1$ CORRECT

**e.** $\pi$

$$\int_0^{\pi/2} \sin(2x)\,dx = \left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/2} = \left[-\frac{1}{2}\cos(\pi)\right] - \left[-\frac{1}{2}\cos(0)\right] = \frac{1}{2} + \frac{1}{2} = 1$$

**13.** Compute $\int_{e^9}^{e^{81}} \frac{1}{x\sqrt{\ln x}}\,dx$

**a.** $2\sqrt{72}$

**b.** $2\sqrt{6}$

**c.** $\sqrt{72}$

**d.** $\sqrt{6}$

**e.** $12$ CORRECT

$$u = \ln x \qquad du = \frac{1}{x}dx$$

$$\int_{e^9}^{e^{81}} \frac{1}{x\sqrt{\ln x}}\,dx = \int_9^{81} \frac{1}{\sqrt{u}}\,du = \left[2\sqrt{u}\right]_9^{81} = 2\sqrt{81} - 2\sqrt{9} = 12$$

Work Out: (10 points each. Part credit possible. Calculators allowed. Show all work.)

**14.** Find the equation of the tangent line to $y = \frac{\ln x}{x^2}$ at $x = e$.

$f(x) = \frac{\ln x}{x^2}$ $\quad f(e) = \frac{\ln e}{e^2} = \frac{1}{e^2}$ $\quad f'(x) = \frac{x^2 \frac{1}{x} - 2x \ln x}{x^4}$ $\quad f'(e) = \frac{e - 2e \ln e}{e^4} = \frac{-1}{e^3}$

$y = f(e) + f'(e)(x - e) = \frac{1}{e^2} - \frac{1}{e^3}(x - e) = -\frac{1}{e^3}x + \frac{2}{e^2}$

**15.** If you start with 4000 bacteria which double every 20 hours, how many bacteria will there be after 30 hours?

$N(t) = 4000e^{kt}$ $\quad 8000 = 4000e^{k20}$ $\quad e^{k20} = 2$ $\quad k = \frac{1}{20}\ln 2$

$N(t) = 4000e^{t(\ln 2)/20}$ $\quad N(30) = 4000e^{30(\ln 2)/20} = 4000 \cdot 2^{3/2} = 4000 \cdot 2\sqrt{2} = 8000\sqrt{2}$

OR $\quad N(t) = 4000(2)^{t/20}$ $\quad N(30) = 4000(2)^{30/20} = 4000 \cdot 2\sqrt{2} = 8000\sqrt{2}$

**16.** In an ideal gas, the pressure $P$, volume $V$ and the absolute temperature $T$ are related by the equation $PV = kT$ where $k$ is a constant. At present $P = 1$ atm, $V = 1000$ liter and $T = 275°$K.

**a.** (5 points) If the volume is held constant and the temperature increases at the rate $\frac{dT}{dt} = \frac{2°\text{K}}{\text{hr}}$, does the pressure increase or decrease and at what rate?

$k = \frac{PV}{T} = \frac{1 \cdot 1000}{275} = \frac{1000}{275}$

$P'V = kT' \quad \Rightarrow \quad P' = \frac{kT'}{V} = \frac{1000}{275}\frac{2}{1000} = \frac{2}{275} \approx 7.3 \times 10^{-3}$ increasing

**b.** (5 points) If the temperature is held constant and the volume increases at the rate $\frac{dV}{dt} = \frac{10 \text{ liter}}{\text{hr}}$, does the pressure increase or decrease and at what rate?

$P'V + PV' = 0 \quad \Rightarrow \quad P' = \frac{-PV'}{V} = \frac{-1 \cdot 10}{1000} = \frac{-1}{100} = -.01$ decreasing

**17.** The position of a particle is given by $x = t^3 - 9t^2 + 33t$. Find the minimum **velocity**. Explain why your critical point is an absolute minimum.

The velocity is $v = 3t^2 - 18t + 33$. We minimize this:

$v' = 6t - 18 = 0 \quad \Rightarrow \quad t = 3 \quad \Rightarrow \quad v = 3(3)^2 - 18(3) + 33 = 6$

$v'' = 6 > 0$. So $v$ is everywhere concave up and $t = 3$ is a local and absolute minimum.

OR $v' = 6(t-3) < 0$ for $t < 3$ and $v' = 6(t-3) > 0$ for $t > 3$. So $v$ is decreasing to the left of $3$ and increasing to the right of $3$ So $t = 3$ is an absolute minimum.

OR $v = 3t^2 - 18t + 33$ is a parabola opening up and so $t = 3$ is an absolute minimum.

**18.** Use the Method of Riemann Sums with equal intervals and Right Endpoints to compute the integral $\int_2^4 3x(x-2)\,dx$.

Use the F.T.C. only to check your answer.

Hints: $\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$\Delta x = \frac{4-2}{n} = \frac{2}{n} \qquad x_i = 2 + i\Delta x = 2 + \frac{2i}{n}$

$f(x) = 3x(x-2) \qquad f(x_i) = 3\left(2 + \frac{2i}{n}\right)\frac{2i}{n} = \frac{12i}{n} + \frac{12i^2}{n^2}$

$$\int_2^4 3x(x-2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{12i}{n} + \frac{12i^2}{n^2}\right)\frac{2}{n} = \lim_{n\to\infty}\left(\frac{24}{n^2}\sum_{i=1}^{n} i + \frac{24}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{24}{n^2}\frac{n(n+1)}{2} + \frac{24}{n^3}\frac{n(n+1)(2n+1)}{6}\right) = \lim_{n\to\infty}\left(12\frac{(n+1)}{n} + 4\frac{(n+1)(2n+1)}{n^2}\right)$$

$$= 12 + 8 = 20$$

Check: $\int_2^4 3x(x-2)\,dx = [x^3 - 3x^2]_2^4 = [64 - 48] - [8 - 12] = 20.$