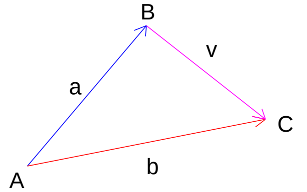
Fall 2004 Math 151
Exam 1A: Solutions
Mon, 04/Oct ©2004, Art Belmonte

1. (a) We have
$2\mathbf{a} - 3\mathbf{b} = 2\,[2, 3] - 3\,[1, -1] = [4, 6] - [3, -3] = [1, 9]$.

2. (c) From the figure below, we have $\mathbf{a} + \mathbf{v} = \mathbf{b}$, from which we conclude $\mathbf{v} = \mathbf{b} - \mathbf{a}$.



3. (d) With $\mathbf{a} = [-2, 3]$ and $\mathbf{b} = [1, 2]$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} = \frac{-2+6}{\sqrt{4+9}} = \frac{4}{\sqrt{13}}.$$

4. (a) Since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, we have

$$\begin{aligned} \mathbf{v} \cdot (2\mathbf{u} - 3\mathbf{v}) &= 2\,(\mathbf{v} \cdot \mathbf{u}) - 3\,(\mathbf{v} \cdot \mathbf{v}) \\ &= 2\,\|\mathbf{v}\|\,\|\mathbf{u}\| \cos 60^\circ - 3\,\|\mathbf{v}\|\,\|\mathbf{v}\| \cos 0^\circ \\ &= 2\,(1)\,(1)\left(\tfrac{1}{2}\right) - 3\,(1)\,(1)\,(1) \\ &= 1 - 3 = -2. \end{aligned}$$

5. (a) For $0 \le t \le \frac{\pi}{2}$, we have $x = \sin t$ and $y = \cos^2 t$. Thus

$$\begin{aligned} \sin^2 t + \cos^2 t &= 1 \\ x^2 + y &= 1 \\ y &= 1 - x^2. \end{aligned}$$

This is part of a parabola.

6. (d) With f defined on an open interval containing 2 and $f(2) = 3$, it is always true that if $\lim_{x \to 2} f(x) = 3 = f(2)$, then f is continuous at $x = 2$.

7. (c) Let $f(c) = c^3 + c - 1 - \pi^2$. Then $f(2) = 9 - \pi^2 < 0$ and $f(3) = 29 - \pi^2 > 0$. Now f is a polynomial and thus continuous everywhere. Therefore, by the Intermediate Value Theorem $f(c) = 0$ for some $c \in (2, 3)$. So for some number $c \in (2, 3)$, we have $c^3 + c - 1 = \pi^2$.

8. (c) Let's divide numerator and denominator of the limiting expression by $x^2 = \sqrt{x^4}$. Then

$$\lim_{x \to \infty} \frac{\sqrt{2x^4 + 3x^2 + 1}}{3x^2 + 2x + 4} = \lim_{x \to \infty} \frac{\sqrt{2 + \dfrac{3}{x^2} + \dfrac{1}{x^4}}}{3 + \dfrac{2}{x} + \dfrac{4}{x^2}} = \frac{\sqrt{2}}{3}.$$

9. (b) Note that $y = \dfrac{x-2}{x^2-4} = \dfrac{(x-2)}{(x-2)\,(x+2)}$. Therefore, *candidates* for vertical asymptotes are $x = 2$ and $x = -2$. Now comes the election!

- As $x \to 2$, we have

$$y = \frac{(x-2)}{(x-2)\,(x+2)} = \frac{1}{x+2} \to \frac{1}{4} \ne \pm\infty.$$

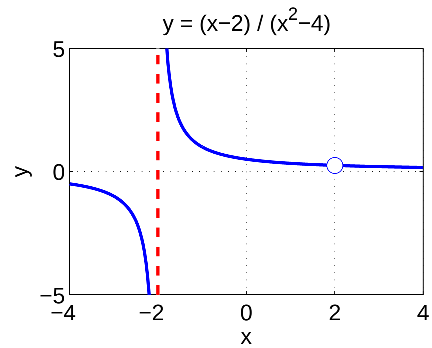
  Thus $x = 2$ is *not* a vertical asymptote.

- As $x \to -2^+$, we see that

$$y = \frac{(x-2)}{(x-2)\,(x+2)} = \frac{1}{x+2} \to \frac{1}{0^+} = +\infty.$$

  Hence $x = -2$ is a vertical asymptote.

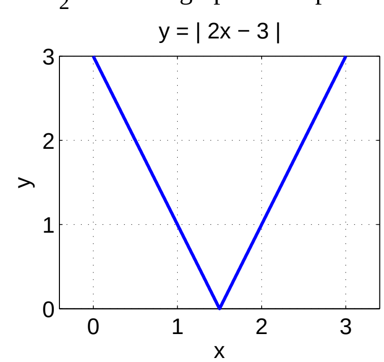
- Here is a plot which corroborates these assertions.



10. (c) Resolve the absolute value.

$$\begin{aligned} f(x) = |2x - 3| &= \begin{cases} 2x - 3, & 2x - 3 \ge 0; \\ -\,(2x - 3)\,, & 2x - 3 < 0 \end{cases} \\ &= \begin{cases} 2x - 3, & x \ge 3/2; \\ 3 - 2x, & x < 3/2 \end{cases} \end{aligned}$$

Now draw a rough sketch to see that $f'(x)$ does not exist for $x = \frac{3}{2}$ since the graph is sharp or kinked thereat.



11. (b) Use the quotient rule to differentiate $f(x) = \dfrac{2x-1}{x^2+1}$.

$$\begin{aligned} f'(x) &= \frac{\left(x^2+1\right)(2) - (2x-1)(2x)}{\left(x^2+1\right)^2} \\ &= \frac{2x^2+2-4x^2+2x}{\left(x^2+1\right)^2} \\ &= \frac{2\left(1+x-x^2\right)}{\left(x^2+1\right)^2} \end{aligned}$$

12. (b) Velocity is the derivative of position.

$$v(t) = s'(t) = 2t+1$$

When $t = 2$ s, the velocity is $v(2) = 5$ m/s.

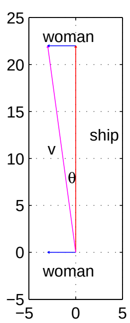
13. (a) With $f(0) = 0$ and $\lim_{x\to 0} \dfrac{f(x)}{x} = -1$, we have $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x-0} = \lim_{x\to 0} \dfrac{f(x)}{x} = -1$; that is, $f'(0) = -1$. Since $g(x) = (2x-1)\, f(x)$, we have

$$\begin{aligned} g'(x) &= (2)\, f(x) + (2x-1)\, f'(x) \\ g'(0) &= (2)\, f(0) + (2(0)-1)\, f'(0) \\ &= (2)\,(0) + (-1)\,(-1) = 1. \end{aligned}$$

14. Let the positive x-axis point east and the positive y-axis point north. Then the velocity of the woman relative to the water is $\mathbf{v} = -3\mathbf{i} + 22\mathbf{j}$, with components in mi/h. The woman's speed is the magnitude of the velocity.

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 22^2} = \sqrt{493} \approx 22.20 \text{ mi/h}$$

Her direction θ points west of north into Quadrant 2 and satisfies $\tan\theta = \frac{3}{22}$, whence $\theta \approx 8°$ or N8°W (from the north, 8° toward the west).



15. The vector from $A(1,1)$ to $B(2,-1)$ is

$$\mathbf{w} = \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [1, -2]\,.$$

A vector perpendicular to $\mathbf{w}$ is $\mathbf{v} = \mathbf{w}^{\perp} = [2, 1]$, a direction vector for our line. A vector equation of the line through $P(3,2)$ in this direction is

$$\begin{aligned} \mathbf{L}(t) &= \overrightarrow{P} + t\mathbf{v} \\ &= [3,2] + t\,[2,1] \\ &= [2t+3,\ t+2]\,. \end{aligned}$$

16. Let $f(x) = \begin{cases} 2c^2x^2 + cx + c, & x < 1; \\ 1, & x = 1; \\ cx+1, & x > 1. \end{cases}$ In order for $\lim_{x\to 1} f(x)$ to exist, we must ensure that the left-hand and right-hand limits are equal: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$.

- From the left we have $\lim_{x\to 1^-} f(x) = 2c^2 + 2c$, whereas $\lim_{x\to 1^+} f(x) = c+1$ for the right-hand limit. Set these one-sided limits equal: $2c^2 + 2c = c + 1$.
- Equivalently, $2c^2 + c - 1 = 0$. Now,

$$(2c-1)\,(c+1) = 0$$

whence $c = -1, \frac{1}{2}$.

17. (i) The derivative of f at a is defined by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

or, equivalently,

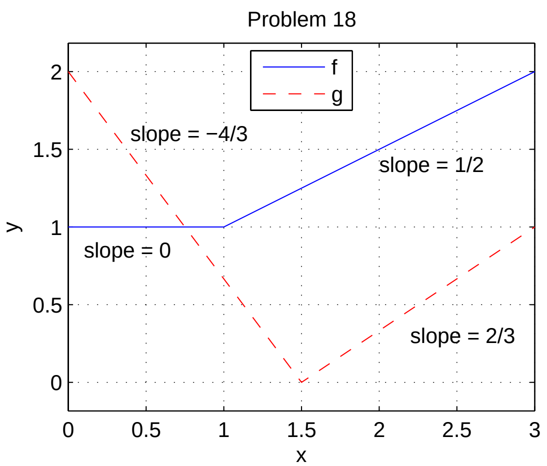
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

(ii) With $f(x) = \dfrac{1}{2x+3}$, we have

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} \\ &= \lim_{x\to 1} \frac{\dfrac{1}{2x+3} - \dfrac{1}{5}}{x-1} \\ &= \lim_{x\to 1} \left(\frac{5-(2x+3)}{5\,(2x+3)}\ \frac{1}{x-1} \right) \\ &= \lim_{x\to 1} \frac{2-2x}{5\,(2x+3)\,(x-1)} \\ &= \lim_{x\to 1} \frac{-2\,(x-1)}{5\,(2x+3)\,(x-1)} \\ &= \lim_{x\to 1} \frac{-2}{5\,(2x+3)} = -\frac{2}{25}. \end{aligned}$$

(iii) A point on the tangent line to the curve $y = \dfrac{1}{2x+3}$ at $x = 1$ is $(1, f(1)) = \left(1, \frac{1}{5}\right)$. The slope of the tangent line from part (a) is $f'(1) = -\frac{2}{25}$. Hence an equation of the tangent line is $y - \frac{1}{5} = -\frac{2}{25}\,(x-1)$ or $y = -\frac{2}{25}x + \frac{7}{25}$.

18. (i) Here is a sketch of the graphs of f and g on the same figure. We label the slopes of the piecewise-linear components.



(ii) Clearly f is not differentiable at $x = 1$ due to the sharp kink in its graph thereat.

(iii) Similarly, g is not differentiable at $x = \frac{3}{2}$ for the same reason.

(iv) At $x = \frac{1}{2}$, we have
$(fg)' = f'g + fg' = (0)\left(g(\frac{1}{2})\right) + (1)\left(-\frac{4}{3}\right) = -\frac{4}{3}$.

(v) At $x = 2$, we have

$$\begin{aligned}\left(\frac{f}{f+g}\right)' &= \frac{(f+g)\,f' - f\,(f'+g')}{(f+g)^2} \\ &= \frac{\left(\frac{3}{2}+\frac{1}{3}\right)\left(\frac{1}{2}\right) - \left(\frac{3}{2}\right)\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{2}+\frac{1}{3}\right)^2} \\ &= -\frac{30}{121} \approx -0.2479.\end{aligned}$$