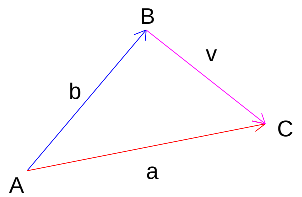
Fall 2004 Math 151
Exam 1B: Solutions
Mon, 04/Oct

1. (b) We have
$3\mathbf{a} - 2\mathbf{b} = 3\,[2, 3] - 2\,[1, -1] = [6, 9] - [2, -2] = [4, 11]$.

2. (b) From the figure below, we have $\mathbf{b} + \mathbf{v} = \mathbf{a}$, from which we conclude $\mathbf{v} = \mathbf{a} - \mathbf{b}$.



3. (b) With $\mathbf{a} = [2, 3]$ and $\mathbf{b} = [1, 2]$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \frac{2+6}{\sqrt{4+9}} = \frac{8}{\sqrt{13}}.$$

4. (e) Since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, we have

$$\begin{aligned}
\mathbf{u}\cdot(3\mathbf{u} - 2\mathbf{v}) &= 3\,(\mathbf{u}\cdot\mathbf{u}) - 2\,(\mathbf{u}\cdot\mathbf{v}) \\
&= 3\,\|\mathbf{u}\|\,\|\mathbf{u}\|\cos 0^\circ - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos 60^\circ \\
&= 3\,(1)\,(1)\,(1) - 2\,(1)\,(1)\left(\tfrac{1}{2}\right) \\
&= 3 - 1 = 2.
\end{aligned}$$

5. (b) For $0 \le t \le \frac{\pi}{2}$, we have $x = \cos t$ and $y = \sin^2 t$. Thus

$$\begin{aligned}
\sin^2 t + \cos^2 t &= 1 \\
y + x^2 &= 1 \\
y &= 1 - x^2.
\end{aligned}$$

This is part of a parabola.

6. (e) With f defined on an open interval containing 2 and $f(2) = 3$, it is always true that if $\lim_{x\to 2} f(x) = 3 = f(2)$, then f is continuous at $x = 2$.

7. (b) Let $f(c) = c^3 + c - 1 - \pi^2$. Then $f(2) = 9 - \pi^2 < 0$ and $f(3) = 29 - \pi^2 > 0$. Now f is a polynomial and thus continuous everywhere. Therefore, by the Intermediate Value Theorem $f(c) = 0$ for some $c \in (2, 3)$. So for some number $c \in (2, 3)$, we have $c^3 + c - 1 = \pi^2$.

8. (d) Let's divide numerator and denominator of the limiting expression by $x^2 = \sqrt{x^4}$. Then

$$\lim_{x\to\infty} \frac{3x^2 + 2x + 4}{\sqrt{2x^4 + 3x^2 + 1}} = \lim_{x\to\infty} \frac{3 + \frac{2}{x} + \frac{4}{x^2}}{\sqrt{2 + \frac{3}{x^2} + \frac{1}{x^4}}} = \frac{3}{\sqrt{2}}.$$

9. (c) Note that $y = \dfrac{x+2}{x^2-4} = \dfrac{(x+2)}{(x-2)\,(x+2)}$. Therefore, *candidates* for vertical asymptotes are $x = 2$ and $x = -2$. Now comes the election!

- As $x \to 2^+$, we have

$$y = \frac{(x+2)}{(x-2)\,(x+2)} = \frac{1}{x-2} \to \frac{1}{0^+} = +\infty.$$

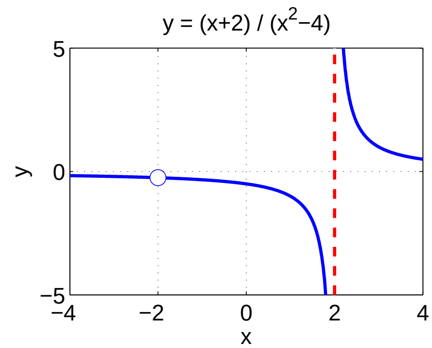
Thus $x = 2$ is a vertical asymptote.

- As $x \to -2$, we see that

$$y = \frac{(x+2)}{(x-2)\,(x+2)} = \frac{1}{x-2} \to -\frac{1}{4} \ne \pm\infty.$$

Hence $x = -2$ is *not* a vertical asymptote.

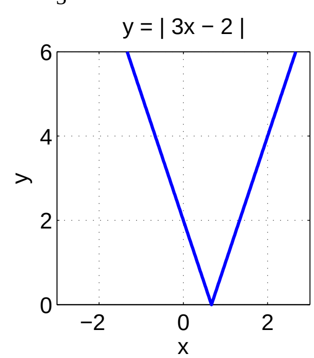
- Here is a plot which corroborates these assertions.



10. (b) Resolve the absolute value.

$$\begin{aligned}
f(x) = |3x - 2| &= \begin{cases} 3x - 2, & 3x - 2 \ge 0; \\ -(3x - 2), & 3x - 2 < 0 \end{cases} \\
&= \begin{cases} 3x - 2, & x \ge 2/3; \\ 2 - 3x, & x < 2/3 \end{cases}
\end{aligned}$$

Now draw a rough sketch to see that $f'(x)$ does not exist for $x = \frac{2}{3}$ since the graph is sharp or kinked thereat.



11. (b) Use the quotient rule to differentiate $f(x) = \dfrac{2x+1}{x^2+1}$.

$$\begin{aligned} f'(x) &= \frac{\left(x^2+1\right)(2) - (2x+1)(2x)}{\left(x^2+1\right)^2} \\ &= \frac{2x^2+2-4x^2-2x}{\left(x^2+1\right)^2} \\ &= \frac{2\left(1-x-x^2\right)}{\left(x^2+1\right)^2} \end{aligned}$$

12. (d) Velocity is the derivative of position.

$$v(t) = s'(t) = 2t+1$$

When $t = 3$ s, the velocity is $v(3) = 7$ m/s.

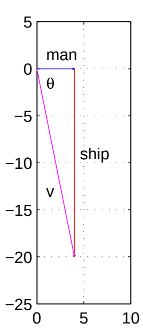
13. (a) With $f(0) = 0$ and $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = -1$, we have $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{f(x)}{x} = -1$; that is, $f'(0) = -1$. Since $g(x) = (2x-1)\, f(x)$, we have

$$\begin{aligned} g'(x) &= (2)\, f(x) + (2x-1)\, f'(x) \\ g'(0) &= (2)\, f(0) + (2(0)-1)\, f'(0) \\ &= (2)(0) + (-1)(-1) = 1. \end{aligned}$$

14. Let the positive x-axis point east and the positive y-axis point north. Then the velocity of the man relative to the water is $\mathbf{v} = 4\mathbf{i} - 20\mathbf{j}$, with components in mi/h. The man's speed is the magnitude of the velocity.

$$\|\mathbf{v}\| = \sqrt{(4)^2 + (-20)^2} = \sqrt{416} \approx 20.40 \text{ mi/h}$$

His direction θ points south of east into Quadrant 4 and satisfies $\tan\theta = \frac{20}{4}$, whence $\theta \approx 79°$ or E79°S (from the east, 79° toward the south) or S11°E (from the south, 11° toward the east).



15. The vector from $A(-2,1)$ to $B(1,-1)$ is

$$\mathbf{w} = \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [3,-2]\,.$$

A vector perpendicular to $\mathbf{w}$ is $\mathbf{v} = \mathbf{w}^{\perp} = [2,3]$, a direction vector for our line. A vector equation of the line through $P(1,3)$ in this direction is

$$\begin{aligned} \mathbf{L}(t) &= \overrightarrow{P} + t\mathbf{v} \\ &= [1,3] + t\,[2,3] \\ &= [2t+1,\ 3t+3]\,. \end{aligned}$$

16. Let $f(x) = \begin{cases} cx+1, & x<1; \\ 1, & x=1; \\ 2c^2x^2+cx+c, & x>1. \end{cases}$ In order for $\lim\limits_{x\to 1} f(x)$ to exist, we must ensure that the left-hand and right-hand limits are equal: $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$.

- From the left we have $\lim\limits_{x\to 1^-} f(x) = c+1$, whereas $\lim\limits_{x\to 1^+} f(x) = 2c^2+2c$ for the right-hand limit. Set these one-sided limits equal: $c+1 = 2c^2+2c$.
- Equivalently, $2c^2+c-1=0$. Now,

$$(2c-1)(c+1) = 0$$

whence $c = -1, \frac{1}{2}$.

17. (i) The derivative of f at a is defined by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

or, equivalently,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

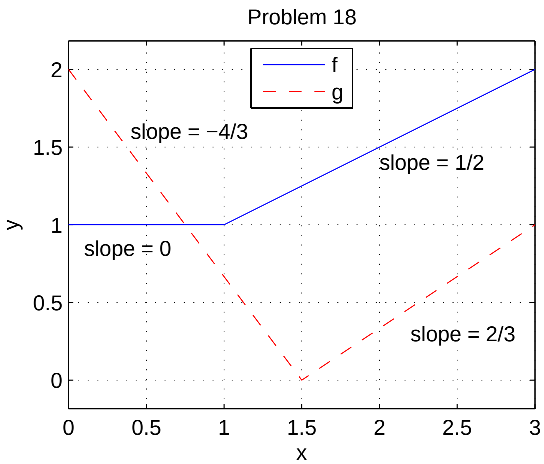
(ii) With $f(x) = \dfrac{1}{3x+2}$, we have

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} \\ &= \lim_{x\to 1} \frac{\frac{1}{3x+2} - \frac{1}{5}}{x-1} \\ &= \lim_{x\to 1} \left(\frac{5-(3x+2)}{5(3x+2)} \frac{1}{x-1} \right) \\ &= \lim_{x\to 1} \frac{3-3x}{5(3x+2)(x-1)} \\ &= \lim_{x\to 1} \frac{-3(x-1)}{5(3x+2)(x-1)} \\ &= \lim_{x\to 1} \frac{-3}{5(3x+2)} = -\frac{3}{25}. \end{aligned}$$

(iii) A point on the tangent line to the curve $y = \dfrac{1}{3x+2}$ at $x = 1$ is $(1, f(1)) = \left(1, \frac{1}{5}\right)$. The slope of the tangent

line from part (a) is $f'(1) = -\frac{3}{25}$. Hence an equation of the tangent line is $y - \frac{1}{5} = -\frac{3}{25}(x-1)$ or
$y = -\frac{3}{25}x + \frac{8}{25}$.

18. (i) Here is a sketch of the graphs of f and g on the same figure. We label the slopes of the piecewise-linear components.



(ii) Clearly g is not differentiable at $x = \frac{3}{2}$ due to the sharp kink in its graph thereat.

(iii) Similarly, f is not differentiable at $x = 1$ for the same reason.

(iv) At $x = \frac{3}{4}$, we have
$(fg)' = f'g + fg' = (0)\left(g(\frac{3}{4})\right) + (1)\left(-\frac{4}{3}\right) = -\frac{4}{3}$.

(v) At $x = 2$, we have

$$\begin{aligned}\left(\frac{g}{f+g}\right)' &= \frac{(f+g)g' - g\left(f'+g'\right)}{(f+g)^2} \\ &= \frac{\left(\frac{3}{2}+\frac{1}{3}\right)\left(\frac{2}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{2}+\frac{1}{3}\right)^2} \\ &= \frac{30}{121} \approx 0.2479.\end{aligned}$$