Fall 2004 Math 151
Exam 2A: Solutions
Mon, 01/Nov ©2004, Art Belmonte

1. (c) As $x \to \infty$, we see that $3^{-x} = \dfrac{1}{3^x} \to 0$ since the numerator is fixed and the denominator increases without bound.

2. (d) The natural logarithm is defined for all positive real arguments; i.e., for $|x| > 0$ or $x \neq 0$.

3. (e) Isolate x step by step.

$$\begin{aligned} \log_2(2x+3) &= 3 \\ 2x+3 &= 2^3 = 8 \\ 2x &= 5 \\ x &= 5/2 \end{aligned}$$

4. (a) The range of f^{-1} is the domain of f. Since we are given $f(x) = \sqrt{1-x}$, we require $1 - x \geq 0$ or $1 \geq x$. This is the interval $(-\infty, 1]$.

5. (a) Since $f(1) = 2$ and $g = f^{-1}$, we have $g(2) = 1$ and thus

$$\begin{aligned} g'(2) &= \frac{1}{f'(g(2))} \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{\left(5x^4 + 3x^2\right)\Big|_{x=1}} = \frac{1}{8}. \end{aligned}$$

6. (b) Use the product rule to differentiate $f(x) = x^2 \tan x$.

$$f'(x) = 2x \tan x + x^2 \sec^2 x$$

7. (b) Now $f(x) = 2\sqrt{e^x} = 2\left(e^x\right)^{1/2} = 2e^{x/2}$, whence

$$f'(x) = 2e^{x/2}\left(\tfrac{1}{2}\right) = e^{x/2} = \left(e^x\right)^{1/2} = \sqrt{e^x}.$$

8. (c) The derivative of position is velocity. In turn the derivative of velocity is acceleration.

$$\begin{aligned} \mathbf{r}(t) &= \left[(\cos t)^2, t\right] \\ \mathbf{v}(t) = \mathbf{r}'(t) &= [2(\cos t)(-\sin t), 1] = [-\sin 2t, 1] \\ \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) &= [-2\cos 2t, 0] \\ &= \left[-2\cos\tfrac{\pi}{3}, 0\right] = [-1, 0] \end{aligned}$$

when $t = \frac{\pi}{6}$.

9. (a) Repeatedly differentiate until a pattern is evident.

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

We see that the derivatives repeat in a cycle of four. Hence $f^{(28)}(x) = \sin x$. More precisely, $f^{(k)} = f^{(k \bmod 4)}$ where $k \bmod 4$ signifies remainder upon division by 4. Because $28 \bmod 4 = 0$, we conclude that

$$f^{(28)}(x) = f^{(0)}(x) = f(x) = \sin x.$$

10. (c) At $x = \sqrt{\pi}$, we have

$$\begin{aligned} f(x) &= \sin\left(x^2\right) = \sin\pi = 0, \\ f'(x) &= \cos\left(x^2\right) \cdot (2x) = -2\sqrt{\pi}. \end{aligned}$$

Therefore, the desired linear approximation is

$$\begin{aligned} L(x) &= f(\sqrt{\pi}) + f'(\sqrt{\pi})(x - \sqrt{\pi}) \\ &= 0 - 2\sqrt{\pi}\left(x - \sqrt{\pi}\right) \\ &= 2\pi - 2\sqrt{\pi}x. \end{aligned}$$

11. (e) The distance between $(x, 0)$ and $(0, 1)$ is

$$s = \sqrt{(x-0)^2 + (0-1)^2} = \left(x^2+1\right)^{1/2}.$$

Differentiation with respect to time t yields

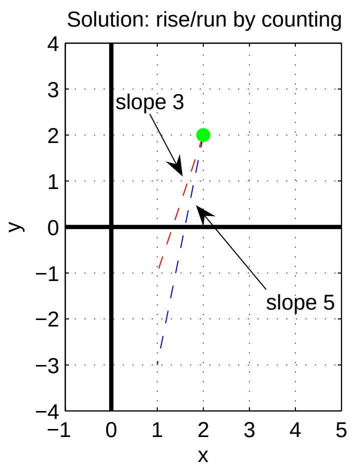
$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{2}\left(x^2+1\right)^{-1/2}\left(2x\,\frac{dx}{dt}\right) \\ &= \frac{2x}{\sqrt{x^2+1}}, \end{aligned}$$

since $dx/dt = 2$ from the statement of the problem.

12. (d) Let $w = 2^{-x}$ (to give it a name). Then

$$2^{-x} + 2^{-x} = w + w = 2w = \left(2^1\right)\left(2^{-x}\right) = 2^{1-x}.$$

13. (c) Geometrically, Newton's method amounts to following tangent lines to where they intersect the x-axis. With the point $P(2, 2)$ on the graph of f and the slope of the tangent line thereat between 3 and 5, we have the following diagram. It is clear from the picture that the next Newton iterate lies in the interval $1 < x < 2$ or $(1, 2)$. No computation is required!



Alternatively, take an analytical approach. Since the graph of $y = f(x)$ passes through $(2, 2)$, we have $f(2) = 2$. Using a slope of $f'(2) = 3$, the next Newton iterate is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2}{3} = \frac{4}{3} \approx 1.33 \in (1, 2)\,.$$

Similarly, using a slope of $f'(2) = 5$ gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2}{5} = \frac{8}{5} = 1.6 \in (1, 2)\,.$$

14. Implicitly differentiate the equation

$$\left(x^2 + y^2\right)^2 = 4\left(x^2 - y^2\right) + 3x + 7$$

with respect to x, then solve for dy/dx.

$$\begin{aligned}
2\left(x^2+y^2\right)\left(2x + 2y\frac{dy}{dx}\right) &= 4\left(2x - 2y\frac{dy}{dx}\right) + 3 \\
\left(x^2+y^2\right)\left(x + y\frac{dy}{dx}\right) &= \left(2x - 2y\frac{dy}{dx}\right) + \frac{3}{4} \\
\left(2y + \left(x^2+y^2\right)y\right)\frac{dy}{dx} &= 2x + \frac{3}{4} - \left(x^2+y^2\right)x \\
\frac{dy}{dx} &= \frac{2x + \frac{3}{4} - \left(x^2+y^2\right)x}{2y + \left(x^2+y^2\right)y}
\end{aligned}$$

Finally, plug in $(x, y) = (2, 1)$.

$$\frac{dy}{dx} = \frac{4\frac{3}{4} - 10}{7} = \frac{\frac{19}{4} - \frac{40}{4}}{7} = \frac{-\frac{21}{4}}{7} = -\frac{3}{4}$$

15. We are given $x = (t+1)^{2/3}$ and $y = te^{-t}$. The point $(x, y) = (1, 0)$ corresponds to $t = 0$.

(i) The parametric derivatives are

$$\begin{aligned}
\frac{dx}{dt} &= \tfrac{2}{3}(t+1)^{-1/3} \\
\frac{dy}{dt} &= (1)\,e^{-t} + te^{-t}(-1) = (1-t)\,e^{-t}.
\end{aligned}$$

(ii) Therefore,

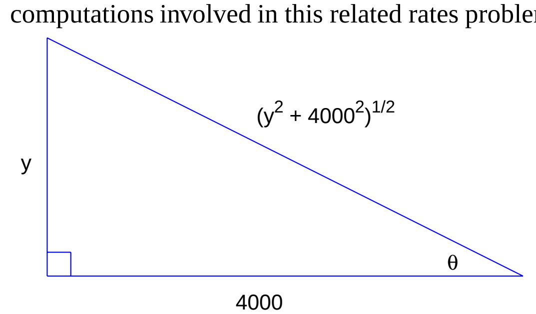
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1-t)\,e^{-t}}{\frac{2}{3}(t+1)^{-1/3}}.$$

(iii) The slope m of the desired tangent line is obtained by plugging $t = 0$ into the expression for dy/dx. This yields $m = \frac{3}{2}$. Hence an equation of the tangent line is $y - 0 = \frac{3}{2}(x-1)$ or $y = \frac{3}{2}x - \frac{3}{2}$.

16. Given $f(x) = e^{ax}\sin bx$, we have

$$\begin{aligned}
f'(x) &= \left(ae^{ax}\right)\sin bx + e^{ax}\left(b\cos bx\right) = e^{ax}\left(a\sin bx + b\cos bx\right) \\
f''(x) &= \left(ae^{ax}\right)\left(a\sin bx + b\cos bx\right) + e^{ax}\left(ab\cos bx - b^2\sin bx\right) \\
&= e^{ax}\left(\left(a^2 - b^2\right)\sin bx + 2ab\cos bx\right)
\end{aligned}$$

17. Let y be the height of the rocket and θ the camera's angle of elevation. Here is a diagram followed by the relevant computations involved in this related rates problem.



$$\begin{aligned}
\tan\theta &= \frac{y}{4000} \\
\sec^2\theta\cdot\frac{d\theta}{dt} &= \frac{dy/dt}{4000} \\
\frac{d\theta}{dt} &= \frac{dy/dt}{4000}(\cos\theta)^2 \\
\frac{d\theta}{dt} &= \frac{600}{4000}\left(\frac{4}{5}\right)^2 = \frac{12}{125} = 0.096
\end{aligned}$$

since for $y = 3000$, we have $\cos\theta = \frac{4000}{\sqrt{3000^2+4000^2}} = \frac{4}{5}$. The camera angle is changing at $\frac{12}{125} = 0.096$ rad/s or $5.5°$/s.

18. The linear approximation to f at $a = 1$ is $L(x) = 2x + 3$. Thus $f(1) = L(1) = 5$ and $f'(1) = L'(1) = 2$. We are given $H = \sqrt{f}$ or $H(x) = (f(x))^{1/2}$. At $x = 1$, $H(x) = \sqrt{5}$ and $H'(x) = \frac{1}{2}(f(x))^{-1/2}f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{5}}\right)(2) = \frac{1}{\sqrt{5}}$. Hence the linear approximation to H at $a = 1$ is

$$L_H(x) = H(1) + H'(1)(x-1) = \sqrt{5} + \frac{1}{\sqrt{5}}(x-1)\,.$$