Fall 2004 Math 151
Exam 3A: Solutions
Mon, 06/Dec ©2004, Art Belmonte

1. (c) We have $f'(x) = \dfrac{1}{2x+4}(2) = \dfrac{2}{2(x+2)} = \dfrac{1}{x+2}$.

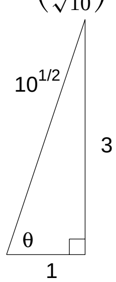
2. (c) The form is $0/0$. Apply L'Hospital's Rule (twice).

$$\begin{aligned}\lim_{x\to 0}\frac{e^{x^2}-1-x^2}{x^3} &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{e^{x^2}(2x)-2x}{3x^2} \\ &= \lim_{x\to 0}\frac{2\left(e^{x^2}-1\right)}{3x} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{2\left(e^{x^2}(2x)\right)}{3} = 0\end{aligned}$$

3. (b) As $x \to 0^+$, we have $\dfrac{\ln x}{\sqrt{x}} \to \dfrac{-\infty}{0^+} = -\infty$. (L'Hospital's Rule is *not* applicable!)

4. (c) We have $\cos^{-1}(\cos(3\pi)) = \cos^{-1}(-1) = \pi$.

5. (e) Let $\theta = \tan^{-1}(3)$. Then $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= \left(\frac{1}{\sqrt{10}}\right)^2 - \left(\frac{3}{\sqrt{10}}\right)^2 = \frac{1}{10} - \frac{9}{10} = -\frac{8}{10} = -\frac{4}{5}$.



6. (d) Expand: $f(x) = \dfrac{x^2+1}{x^2} = 1 + x^{-2}$. Now compute an antiderivative: $F(x) = x - x^{-1} = x - \dfrac{1}{x} = \dfrac{x^2-1}{x}$.

7. (c) Given F is an antiderivative of f, we have $F'(x) = f(x)$. Since $\dfrac{d}{dx}\left(\dfrac{F(3x)}{3}\right) = \dfrac{1}{3}F'(3x)\cdot 3 = F'(3x) = f(3x)$, we see that $\dfrac{F(3x)}{3}$ is an antiderivative of $f(3x)$.

8. (a) Recall that velocity is the derivative of position and acceleration the derivative of velocity. We antidifferentiate acceleration to get velocity, then antidifferentiate velocity to get position, resolving constants along the way.

$$\begin{aligned} v'(t) = a(t) &= 6t \\ v(t) &= 3t^2 + C \\ 1 = v(0) &= 0 + C \\ C &= 1 \\ s'(t) = v(t) &= 3t^2 + 1 \\ s(t) &= t^3 + t + K \\ 1 = s(0) &= 0 + 0 + K \\ K &= 1 \\ s(t) &= t^3 + t + 1 \\ s(1) &= 1 + 1 + 1 = 3 \end{aligned}$$

9. (b) We have $f'(x) = 1 - \dfrac{1}{\sqrt{1-x^2}}$ on $(-1, 1)$. Therefore, $f'(x) = 0$ when $x = 0 \in (-1, 1)$. Apply the Closed Interval Method on $[-1, 1]$. Compute function values of f at the critical value and at the endpoints.

x	$f(x) = x - \sin^{-1} x$	COMMENT
-1	$-1 - \left(-\frac{\pi}{2}\right) \approx 0.57$	absolute maximum
0	$0 + 0 = 0$	—
1	$1 - \frac{\pi}{2} \approx -0.57$	absolute minimum

10. (b) H is increasing where $H' > 0$; that is, on $(-1, 1)$.

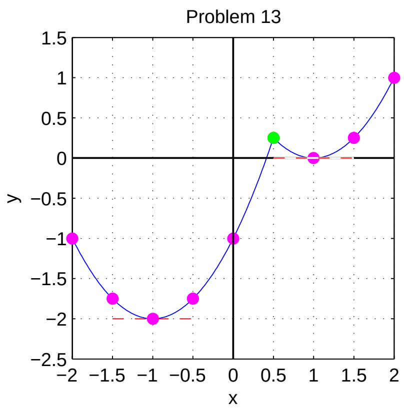
11. (c) Since H' changes sign from $+$ to 0 to $-$ as x increases through $x = 1$, the first derivative test says that H has a local maximum at $x = 1$.

12. (c) Compute $f'(x) = (1)e^x + xe^x = (x+1)e^x$ and $f''(x) = (1)e^x + (x+1)e^x = (x+2)e^x$. Solve $f''(x) = 0$ to obtain $x = -2$. Since f'' changes sign as x increases through -2, the point of inflection is $\left(-2, -2e^{-2}\right)$.

13. (d) Resolve the absolute value, then differentiate.

$$\begin{aligned} f(x) = x^2 - |2x-1| &= \begin{cases} x^2 - (-(2x-1)), & x < \frac{1}{2} \\ x^2 - (2x-1), & x \ge \frac{1}{2} \end{cases} \\ f(x) &= \begin{cases} x^2 + 2x - 1, & x < \frac{1}{2} \\ x^2 - 2x + 1, & x \ge \frac{1}{2} \end{cases} \\ f'(x) &= \begin{cases} 2x + 2, & x < \frac{1}{2} \\ 2x - 2, & x > \frac{1}{2} \end{cases} \end{aligned}$$

(Note: f is not differentiable at $x = \frac{1}{2}$ since its graph is sharp or kinked there. See plot on next page.) Now $f'(x) = 0$ for $x = \pm 1$. Therefore, the critical points of f are $x = -1, \frac{1}{2}, 1$.



14. Since the bacterial culture grows at a rate proportional to its size, it experiences exponential growth. Let y be the number of bacteria at time t. Then $y = y_0 e^{kt} = 1{,}200 e^{kt}$.

- After 2 hours there are 2,500 bacteria.

$$
\begin{aligned}
2{,}500 &= 1{,}200 e^{2k} \\
\tfrac{25}{12} &= e^{2k} \\
\ln \tfrac{25}{12} &= 2k \\
k &= \tfrac{1}{2} \ln \tfrac{25}{12} \\
y &= 1{,}200 e^{\left(\frac{1}{2}\ln\frac{25}{12}\right)t} = 1{,}200 \left(\frac{25}{12}\right)^{t/2}
\end{aligned}
$$

- After 6 hours there are $y = 1{,}200 \left(\dfrac{25}{12}\right)^3 \approx 10{,}851$ bacteria.

15. (i) Recursively apply the Chain Rule.

$$
\begin{aligned}
U'(x) &= \frac{d}{dx}\left(f\left(\tan^{-1}\left(e^{-x}\right)\right)\right) \\
&= f'\left(\tan^{-1}\left(e^{-x}\right)\right) \cdot \frac{1}{1+\left(e^{-x}\right)^2} \cdot e^{-x}(-1) \\
&= -\frac{f'\left(\tan^{-1}\left(e^{-x}\right)\right)}{e^{x}+e^{-x}}
\end{aligned}
$$

(ii) Use logarithmic differentiation.

$$
\begin{aligned}
V(x) &= (2+\cos x)^{f(x)} \\
\ln V(x) &= f(x)\ln(2+\cos x) \\
\frac{1}{V(x)} V'(x) &= f'(x)\ln(2+\cos x) + f(x)\frac{1}{2+\cos x}(-\sin x) \\
V'(x) &= (2+\cos x)^{f(x)}\left(f'(x)\ln(2+\cos x) - \frac{f(x)\sin x}{2+\cos x}\right)
\end{aligned}
$$

16. As $x \to \infty$, the expression $\left(1-\dfrac{1}{2x}\right)^{3x}$ is seen to be an indeterminate power, 1^∞.

- Let $y = \left(1-\dfrac{1}{2x}\right)^{3x} = \left(1-\frac{1}{2}x^{-1}\right)^{3x}$.
- Then $\ln y = 3x \ln\left(1-\frac{1}{2}x^{-1}\right) = \dfrac{3\ln\left(1-\frac{1}{2}x^{-1}\right)}{x^{-1}}$, which is an indeterminate quotient $0/0$ as $x \to \infty$.
- We may therefore apply L'Hospital's Rule.

$$
\begin{aligned}
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{3\ln\left(1-\frac{1}{2}x^{-1}\right)}{x^{-1}} \\
&\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{3\left(\dfrac{1}{1-\frac{1}{2}x^{-1}}\right)\left(\frac{1}{2}x^{-2}\right)}{-x^{-2}} \\
&= \lim_{x\to\infty} \frac{3}{-2}\left(\frac{1}{1-\frac{1}{2x}}\right) = -\frac{3}{2}.
\end{aligned}
$$

- Hence our limit $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{2x}\right)^{3x}$ is given by $\lim y = \lim e^{\ln y} = e^{\lim \ln y} = e^{-3/2}$.

17. (i) The function increases where $f'(x) = \dfrac{4x}{\left(x^2+3\right)^2} > 0$; that is, for $x > 0$ or $(0, \infty)$.

(ii) Next, f is concave down where

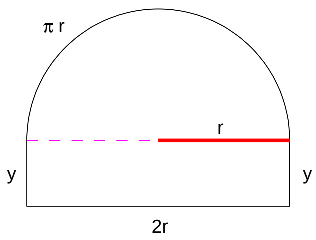
$$
f''(x) = \frac{12-12x^2}{\left(x^2+3\right)^3} < 0;
$$

i.e., for $|x| > 1$ or $(-\infty, -1) \cup (1, \infty)$.

(iii)
- Since f' changes sign from $-$ to 0 to $+$ as x increases through $x = 0$, the first derivative test says that f has a local minimum at $x = 0$.
- Moreover, since $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$, we conclude that f has an absolute minimum at $x = 0$ by the first derivative test for absolute extrema.
- The first derivative of f is defined everywhere from the statement of the problem. Yet nowhere does f' change sign from $+$ to 0 to $-$. Therefore, f has *no* local maximum.
- Since $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$, we conclude that f has *no* absolute maximum. This is because f is always increasing as we go further to the right of $x = 0$ (and the same is true as we go further to the left of $x = 0$).

(iv) The function f has inflection points at $x = \pm 1$ since f'' changes sign there.

18. (i) Let y be the height of the window's rectangular portion and r be the radius of the semicircle. Here is a diagram.



(ii) The perimeter P of the window is 30 feet.

$$\begin{aligned} P &= 30 \\ \tfrac{1}{2}(2\pi r) + 2r + 2y &= 30 \\ (\pi + 2)r + 2y &= 30 \\ y &= \tfrac{1}{2}(30 - (\pi + 2)r) \end{aligned}$$

The area A of the window is

$$\begin{aligned} A &= \tfrac{1}{2}(\pi r^2) + 2ry \\ A &= \tfrac{1}{2}\pi r^2 + r(30 - (\pi + 2)r) \\ A &= 30r - \left(2 + \tfrac{\pi}{2}\right)r^2 \quad \text{[Fast track?} \\ \text{domain} &: \ 0 \le r \le \frac{30}{\pi + 2}. \quad \text{See right!}\nearrow] \end{aligned}$$

[NOTE: When $r = \dfrac{30}{\pi + 2}$, we have $y = 0$ and the window is semicircular. There is no rectangular part.]

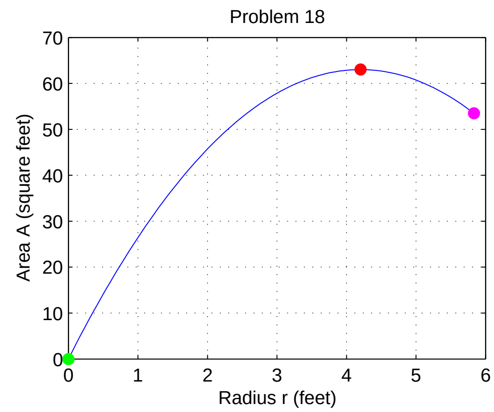
(iii) Use the Closed Interval Method for Absolute Extrema. (That's why we allowed $r = 0$ above. A window having width zero has area zero.)

$$\begin{aligned} A' &= 30 - (4 + \pi)r = 0 \\ r &= \frac{30}{\pi + 4} \approx 4.2 \text{ ft} \end{aligned}$$

- Crank out function values of A at this interior r-value and at the endpoints of the domain of A.

r	A	COMMENT
0	0	No width: zero area!
$\dfrac{30}{\pi+4}$	$\dfrac{450}{\pi+4}$	Maximum area
$\dfrac{30}{\pi+2}$	$\dfrac{450\pi}{(\pi+2)^2}$	Semicircular region

- Since $\dfrac{450}{\pi+4} \approx 63.01$ and $\dfrac{450\pi}{(\pi+2)^2} \approx 53.48$, we see that the maximum area is $\dfrac{450}{\pi+4} \approx 63 \text{ ft}^2$. This occurs when $r = y = \dfrac{30}{\pi+4} \approx 4.2$ ft. Here is a plot of A versus r.



(iii-alt) Note that $A = 30r - \left(2 + \frac{\pi}{2}\right)r^2$ is quadratic. Its graph is therefore a downward opening parabola (since the coefficient of r^2 is negative). Hence the absolute maximum value of A occurs at the parabola's vertex; i.e., where $A' = 0$.

$$\begin{aligned} A' &= 30 - (4 + \pi)r = 0 \\ r &= \frac{30}{\pi + 4} \approx 4.2 \text{ ft} \end{aligned}$$

For this value of r, we find that y is also equal to $\dfrac{30}{\pi+4} \approx 4.2$ ft.