MATH 151, FALL SEMESTER 2011
COMMON EXAMINATION I - VERSION A - SOLUTIONS

Name (print): ____________________ Instructor's name: ____________________

Signature: ____________________ Section No: ____________________

Part 1 – Multiple Choice (12 questions, 4 points each, No Calculators)

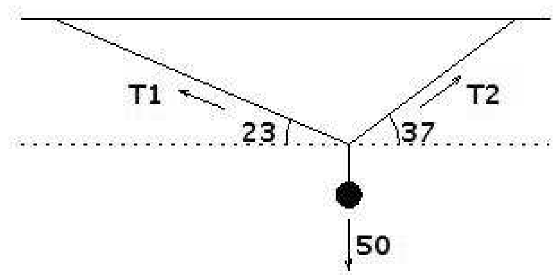
Write your name, section number, and version letter (**A**) of the exam on the ScanTron form.
Mark your responses on the ScanTron form and on the exam itself

1. Let $\mathbf{v} = \langle 2,4 \rangle$ and $\mathbf{w} = -2\mathbf{i} + 6\mathbf{j}$. Compute $\left| \frac{1}{2}\mathbf{v} - \mathbf{w} \right|$.

- **a.** 1
- **b.** 2
- **c.** 3
- **d.** 4
- **e.** 5 Correct Choice

SOLUTION: $\frac{1}{2}\mathbf{v} - \mathbf{w} = \frac{1}{2}\langle 2,4 \rangle - \langle -2,6 \rangle = \langle 1,2 \rangle + \langle 2,-6 \rangle = \langle 3,-4 \rangle$ $\quad \left| \frac{1}{2}\mathbf{v} - \mathbf{w} \right| = \sqrt{3^2 + 4^2} = 5$

2. A ball whose weight is 50 Newtons hangs from two wires, one at angle 23° from horizontal, and the other at angle 37° from horizontal. Let $\mathbf{T}_1$ be the tension in the first wire, and $\mathbf{T}_2$ be the tension in the second wire. Which set of equations can be used to solve for $\mathbf{T}_1$ and $\mathbf{T}_2$?



- **a.** $-|\mathbf{T}_1|\cos 23° + |\mathbf{T}_2|\cos 37° = 0$ and $|\mathbf{T}_1|\sin 23° + |\mathbf{T}_2|\sin 37° = 50$ Correct Choice
- **b.** $-|\mathbf{T}_1|\cos 23° + |\mathbf{T}_2|\cos 37° = 50$ and $-|\mathbf{T}_1|\sin 23° + |\mathbf{T}_2|\sin 37° = 0$
- **c.** $|\mathbf{T}_1|\cos 23° + |\mathbf{T}_2|\cos 37° = 50$ and $-|\mathbf{T}_1|\sin 23° + |\mathbf{T}_2|\sin 37° = 0$
- **d.** $-|\mathbf{T}_1|\cos 23° + |\mathbf{T}_1|\cos 37° = 0$ and $|\mathbf{T}_2|\sin 23° + |\mathbf{T}_2|\sin 37° = 50$
- **e.** $|\mathbf{T}_1|\cos 23° + |\mathbf{T}_2|\cos 37° = 0$ and $-|\mathbf{T}_1|\sin 23° + |\mathbf{T}_2|\sin 37° = 50$

SOLUTION: $\mathbf{T}_1 = \langle -|\mathbf{T}_1|\cos 23°, |\mathbf{T}_1|\sin 23° \rangle$ $\quad \mathbf{T}_2 = \langle |\mathbf{T}_2|\cos 37°, |\mathbf{T}_2|\sin 37° \rangle$ $\quad \mathbf{W} = \langle 0,-50 \rangle$
$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{0}$ $\quad -|\mathbf{T}_1|\cos 23° + |\mathbf{T}_2|\cos 37° = 0$ $\quad |\mathbf{T}_1|\sin 23° + |\mathbf{T}_2|\sin 37° - 50 = 0$

3. Find the angle between the vectors $\mathbf{v} = \langle 1,2 \rangle$ and $\mathbf{w} = \langle 3,1 \rangle$.

- **a.** 0°
- **b.** 30°
- **c.** 45° Correct Choice
- **d.** 60°
- **e.** 90°

SOLUTION: $|\mathbf{v}| = \sqrt{5}$ $\quad |\mathbf{w}| = \sqrt{10}$ $\quad \mathbf{v} \cdot \mathbf{w} = 5$ $\quad \cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{1}{\sqrt{2}}$ $\quad \theta = 45°$

4. Find the scalar projection (component) and vector projection of $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$ onto $\mathbf{w} = 4\mathbf{i} + 3\mathbf{j}$.

- **a.** scalar projection $= \frac{16}{13}$ vector projection $= \frac{64}{169}\mathbf{i} + \frac{48}{169}\mathbf{j}$
- **b.** scalar projection $= -\frac{16}{13}$ vector projection $= -\frac{64}{169}\mathbf{i} - \frac{48}{169}\mathbf{j}$
- **c.** scalar projection $= -\frac{16}{5}$ vector projection $= \frac{64}{25}\mathbf{i} - \frac{48}{25}\mathbf{j}$
- **d.** scalar projection $= \frac{16}{5}$ vector projection $= \frac{64}{25}\mathbf{i} + \frac{48}{25}\mathbf{j}$
- **e.** scalar projection $= -\frac{16}{5}$ vector projection $= -\frac{64}{25}\mathbf{i} - \frac{48}{25}\mathbf{j}$ Correct Choice

SOLUTION:
$\text{comp}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|} = \frac{20-36}{5} = -\frac{16}{5}$ $\quad \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{20-36}{5^2}(4\mathbf{i}+3\mathbf{j}) = -\frac{64}{25}\mathbf{i} - \frac{48}{25}\mathbf{j}$

5. Find the Cartesian equation for the graph of the parametric curve $x = -1 + t$ and $y = t^2 - t$.

- **a.** $y = x^2 - x$
- **b.** $y = x^2 + x$ Correct Choice
- **c.** $y = x^2 + 3x$
- **d.** $y = x^2 + 3x + 2$
- **e.** $y = x^2 - 3x + 2$

SOLUTION: $t = x + 1$ $\quad y = (x+1)^2 - (x+1) = x^2 + x$

6. Find a vector equation for the line which contains the point $(2,-1)$ and is parallel to $\langle 3,4\rangle$.

- **a.** $\mathbf{r}(t) = \langle 1+4t, -2+3t\rangle$
- **b.** $\mathbf{r}(t) = \langle -3-t, -4+2t\rangle$
- **c.** $\mathbf{r}(t) = \langle 3+2t, 4-t\rangle$
- **d.** $\mathbf{r}(t) = \langle 2+3t, -1+4t\rangle$ Correct Choice
- **e.** $\mathbf{r}(t) = \langle -2-3t, 1-4t\rangle$

SOLUTION: $\mathbf{r}_0 = \langle 2,-1\rangle$ $\quad \mathbf{v} = \langle 3,4\rangle$ $\quad \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 2,-1\rangle + t\langle 3,4\rangle = \langle 2+3t, -1+4t\rangle$

7. Let $f(x) = \frac{x^2-4}{(x-2)^2}$. Which of the following is true?

- **a.** $\lim_{x\to 2^-} f(x) = +\infty$ and $\lim_{x\to 2^+} f(x) = +\infty$
- **b.** $\lim_{x\to 2^-} f(x) = +\infty$ and $\lim_{x\to 2^+} f(x) = -\infty$
- **c.** $\lim_{x\to 2^-} f(x) = -\infty$ and $\lim_{x\to 2^+} f(x) = +\infty$ Correct Choice
- **d.** $\lim_{x\to 2^-} f(x) = -\infty$ and $\lim_{x\to 2^+} f(x) = -\infty$
- **e.** None of these.

SOLUTION: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x+2}{x-2} = \frac{4^-}{0^-} = -\infty$ $\quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x+2}{x-2} = \frac{4^+}{0^+} = +\infty$

8. Compute $\lim_{t \to 1} \frac{1 - t^2}{1 - \sqrt{t}}$

a. 1
b. 2
c. 3
d. 4 Correct Choice
e. Does not exist

SOLUTION: $\lim_{t \to 1} \frac{1 - t^2}{1 - \sqrt{t}} = \lim_{t \to 1} \frac{1 - t^2}{1 - \sqrt{t}} \cdot \frac{1 + \sqrt{t}}{1 + \sqrt{t}} = \lim_{t \to 1} \frac{(1 - t^2)(1 + \sqrt{t}\,)}{1 - t} = \lim_{t \to 1}(1 + t)(1 + \sqrt{t}\,) = 4$

9. Which interval contains the unique real solution of the equation $2x^3 + x^2 + 2 = 0$?

a. $(-2, -1)$ Correct Choice
b. $(-1, 0)$
c. $(0, 1)$
d. $(1, 2)$
e. $(2, 3)$

SOLUTION: Let $f(x) = 2x^3 + x^2 + 2$. $f(-2) = -16 + 4 + 2 = -10$ $f(-1) = -2 + 1 + 2 = 1$
Since $-10 < 0 < 1$, by the I.V.T. there is a $c \in (-2, -1)$ where $f(c) = 2c^3 + c^2 + 2 = 0$

10. Which of the following is a horizontal asymptote of $f(x) = \frac{3x^2 + 2}{(x - 2)(x + 2)}$?

a. $y = \frac{1}{3}$
b. $y = 3$ Correct Choice
c. $y = -2$
d. $y = -\frac{1}{2}$
e. None of the above

SOLUTION: $\lim_{x \to \infty} \frac{3x^2 + 2}{(x - 2)(x + 2)} = \lim_{x \to \infty} \frac{3x^2 + 2}{x^2 - 4} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to \infty} \frac{3 + 2/x^2}{1 - 4/x^2} = 3$

11. Evaluate $\lim_{x \to 3^+} \frac{2x^2 - 3x}{x(x + 3)}$

a. $-\infty$
b. 0
c. $\frac{1}{2}$ Correct Choice
d. 1
e. ∞

SOLUTION: $\lim_{x \to 3^+} \frac{2x^2 - 3x}{x(x + 3)} = \frac{2 \cdot 9 - 3 \cdot 3}{3(3 + 3)} = \frac{1}{2}$

12. Evaluate $\lim_{x \to -\infty} \frac{2x^2 + 3x}{x - 3}$

a. $-\infty$ Correct Choice
b. -1
c. $-\frac{2}{3}$
d. 2
e. ∞

SOLUTION: $\lim_{x \to -\infty} \frac{2x^2 + 3x}{x - 3} = \lim_{x \to -\infty} \frac{2x^2 + 3x}{x - 3} \cdot \frac{1/x}{1/x} = \lim_{x \to -\infty} \frac{2x + 3}{1 - 3/x} = -\infty$

Part 2 – Work Out Problems (5 questions. Points indicated. No Calculators)

Solve each problem in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

13. (10 points) An object is moving in the xy-plane and its position vector after t seconds is $\mathbf{r}(t) = \langle t-3, t^2-2t\rangle$.

a. Find the position vector of the object at time $t = 5$.

SOLUTION: $\mathbf{r}(5) = \langle 5-3, 5^2-2\cdot 5\rangle = \langle 2, 15\rangle$

b. Does the object pass through the point $(1,8)$? If yes, when? If no, why not?

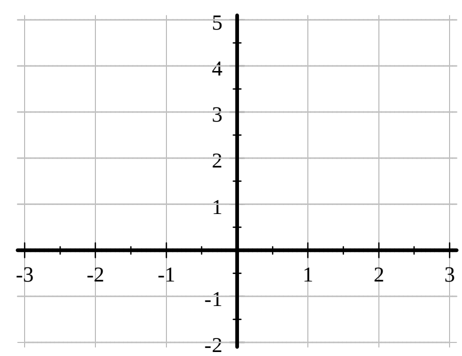
SOLUTION: $\langle t-3, t^2-2t\rangle = \langle 1, 8\rangle \quad t-3=1 \quad t^2-2t=8$
$t = 4$ satisfies both equations. YES

c. Does the object pass through the point $(2,20)$? If yes, when? If no, why not?

SOLUTION: $\langle t-3, t^2-2t\rangle = \langle 2, 20\rangle \quad t-3=2 \quad t^2-2t=20$
$t = 5$ is the only solution of the first equation but does not satisfy the second because $t^2-2t = 5^2-2\cdot 5 = 15 \neq 20$. NO

14. (14 points) Sketch the graph of the function

$$g(x) = \begin{cases} -x-1 & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ |x| & \text{if } -2 < x < 1 \\ x^2-2 & \text{if } x \geq 1 \end{cases}$$



Then determine each of the following

$\lim_{x\to -2^-} g(x) =$ ________ $\lim_{x\to -2^+} g(x) =$ ________ $g(-2) =$ ________

$\lim_{x\to 1^-} g(x) =$ ________ $\lim_{x\to 1^+} g(x) =$ ________ $g(1) =$ ________

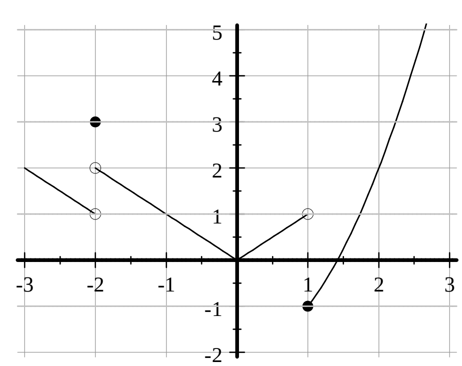
List all value(s) of x where $g(x)$ is NOT differentiable: ____________________

SOLUTION:

$\lim_{x\to -2^-} g(x) = \underline{\ 1\ }$ $\lim_{x\to -2^+} g(x) = \underline{\ 2\ }$ $g(-2) = \underline{\ 3\ }$

$\lim_{x\to 1^-} g(x) = \underline{\ 1\ }$ $\lim_{x\to 1^+} g(x) = \underline{\ -1\ }$ $g(1) = \underline{\ -1\ }$

$g(x)$ is NOT differentiable at : $\underline{\ -2, \quad 0, \quad 1\ }$



15. (9 points) Compute each of the following or prove the limit does not exist.

a. $\lim_{x\to 2^+} \dfrac{|x-2|}{x^2-2x} =$

SOLUTION: When $x > 2$, we have $x - 2 > 0$ and $|x-2| = x-2$. So

$$\lim_{x\to 2^+} \frac{|x-2|}{x^2-2x} = \lim_{x\to 2^+} \frac{x-2}{x(x-2)} = \lim_{x\to 2^+} \frac{1}{x} = \frac{1}{2}$$

b. $\lim_{x\to 2^-} \dfrac{|x-2|}{x^2-2x} =$

SOLUTION: When $x < 2$, we have $x - 2 < 0$ and $|x-2| = -(x-2)$. So

$$\lim_{x\to 2^-} \frac{|x-2|}{x^2-2x} = \lim_{x\to 2^-} \frac{-(x-2)}{x(x-2)} = \lim_{x\to 2^-} \frac{-1}{x} = -\frac{1}{2}$$

c. $\lim_{x\to 2} \dfrac{|x-2|}{x^2-2x} =$

SOLUTION: Since $\lim_{x\to 2^+} \dfrac{|x-2|}{x^2-2x} = \dfrac{1}{2} \neq \lim_{x\to 2^-} \dfrac{|x-2|}{x^2-2x} = -\dfrac{1}{2}$,

the 2-sided limit $\lim_{x\to 2} \dfrac{|x-2|}{x^2-2x}$ does not exist.

16. (9 points) Consider $f(x) = \begin{cases} \dfrac{x^2-2x-8}{x-4} & \text{if } x \neq 4 \\ p & \text{if } x = 4 \end{cases}$

a. Find $\lim_{x\to 4} f(x)$ or explain why it does not exist.

SOLUTION: $\lim_{x\to 4} f(x) = \lim_{x\to 4} \dfrac{x^2-2x-8}{x-4} = \lim_{x\to 4} \dfrac{(x-4)(x+2)}{x-4} = \lim_{x\to 4} (x+2) = 6$

b. Find the value(s) of p that make $f(x)$ continuous at $x = 4$ or explain why no such p exists.

SOLUTION:
For $f(x)$ to be continuous at $x = 4$, we must have $f(4) = \lim_{x\to 4} f(x)$ or $p = 6$.

17. (10 points) Consider the function $f(x) = \dfrac{1}{x}$.

a. Find $f'(x)$, the derivative of $f(x)$, using the limit definition of the derivative.

SOLUTION:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{x-(x+h)}{(x+h)x} = \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

b. Find the slope of the tangent line to the curve $y = f(x)$ at $x = 3$.

SOLUTION: slope $= f'(3) = \dfrac{-1}{3^2} = \dfrac{-1}{9}$