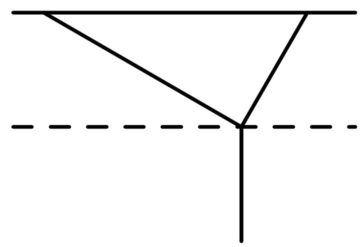
MATH 151, **Fall 2013**

FINAL EXAMINATION - **SOLUTIONS**

Part 1 – Multiple Choice (**14 questions**, **4 points each**, **No Calculators**)

Write your name and section number on the ScanTron form.
Mark your responses on the ScanTron form and on the exam itself

1. A ball whose weight is $F = 50$ Newtons hangs from two wires, one at angle $30°$ above horizontal on the left, and the other at angle $60°$ above horizontal on the right. Let $\vec{T}_1$ be the tension in the first wire, and $\vec{T}_2$ be the tension in the second wire. Which set of equations can be used to solve for $|\vec{T}_1|$ and $|\vec{T}_2|$?



a. $-\frac{\sqrt{3}}{2}|\vec{T}_1| + \frac{1}{2}|\vec{T}_2| = 0 \qquad \frac{1}{2}|\vec{T}_1| + \frac{\sqrt{3}}{2}|\vec{T}_2| = 50$ Correct Choice

b. $-\frac{1}{2}|\vec{T}_1| + \frac{\sqrt{3}}{2}|\vec{T}_2| = 0 \qquad \frac{\sqrt{3}}{2}|\vec{T}_1| + \frac{1}{2}|\vec{T}_2| = 50$

c. $\frac{\sqrt{3}}{2}|\vec{T}_1| + \frac{1}{2}|\vec{T}_2| = 0 \qquad -\frac{1}{2}|\vec{T}_1| + \frac{\sqrt{3}}{2}|\vec{T}_2| = 50$

d. $\frac{1}{2}|\vec{T}_1| + \frac{\sqrt{3}}{2}|\vec{T}_2| = 0 \qquad -\frac{\sqrt{3}}{2}|\vec{T}_1| + \frac{1}{2}|\vec{T}_2| = -50$

e. $-\frac{1}{2}|\vec{T}_1| + \frac{\sqrt{3}}{2}|\vec{T}_2| = 0 \qquad \frac{\sqrt{3}}{2}|\vec{T}_1| + \frac{1}{2}|\vec{T}_2| = -50$

SOLUTION: $\vec{F}_g = -50\,\hat{\jmath}$

$\vec{T}_1 = -|\vec{T}_1|\cos 30°\,\hat{\imath} + |\vec{T}_1|\sin 30°\,\hat{\jmath} = -\frac{\sqrt{3}}{2}|\vec{T}_1|\,\hat{\imath} + \frac{1}{2}|\vec{T}_1|\,\hat{\jmath}$

$\vec{T}_2 = |\vec{T}_2|\cos 60°\,\hat{\imath} + |\vec{T}_2|\sin 60°\,\hat{\jmath} = \frac{1}{2}|\vec{T}_2|\,\hat{\imath} + \frac{\sqrt{3}}{2}|\vec{T}_2|\,\hat{\jmath}$

The equations are the $\hat{\imath}$ and $\hat{\jmath}$ components of the sum.

2. Find the cosine of the angle between the vectors $\vec{A} = \langle 3,4\rangle$ and $\vec{B} = \langle 12,-5\rangle$

a. $\cos(\theta) = \frac{56}{\sqrt{65}}$

b. $\cos(\theta) = \frac{4}{\sqrt{65}}$

c. $\cos(\theta) = \frac{16}{65}$ Correct Choice

d. $\cos(\theta) = \frac{4}{65}$

e. $\cos(\theta) = \frac{56}{65}$

SOLUTION: $\cos(\theta) = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{36-20}{5\cdot 13} = \frac{16}{65}$

3. Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{x^4 - 4x^3 - 5} - \sqrt{x^4 + 4x^3 + 5}}{x}$

a. $-\infty$
b. -8
c. -4 Correct Choice
d. -2
e. 0

SOLUTION: $\lim_{x\to\infty} \dfrac{\sqrt{x^4 - 4x^3 - 5} - \sqrt{x^4 + 4x^3 + 5}}{x} \dfrac{\left(\sqrt{x^4 - 4x^3 - 5} + \sqrt{x^4 + 4x^3 + 5}\right)}{\left(\sqrt{x^4 - 4x^3 - 5} + \sqrt{x^4 + 4x^3 + 5}\right)}$

$= \lim_{x\to\infty} \dfrac{(x^4 - 4x^3 - 5) - (x^4 + 4x^3 + 5)}{x\left(\sqrt{x^4 - 4x^3 - 5} + \sqrt{x^4 + 4x^3 + 5}\right)} = \lim_{x\to\infty} \dfrac{-8x^3 - 10}{x\left(\sqrt{x^4 - 4x^3 - 5} + \sqrt{x^4 + 4x^3 + 5}\right)} \dfrac{x^{-3}}{x^{-3}} = \dfrac{-8}{2} = -4$

4. Which interval contains the unique real solution of the equation $x^3 + 2x = 4$

a. $(-2,-1)$
b. $(-1,0)$
c. $(0,1)$
d. $(1,2)$ Correct Choice

SOLUTION: $f(x) = x^3 + 2x$ $f(0) = 0$ $f(1) = 3$ $f(2) = 12$ 4 is between 3 and 12.

5. Find the 2017-th derivative of $f(x) = x + \ln x$

a. $2014!\, x^{-2015}$
b. $-2016!\, x^{-2017}$
c. $2015!\, x^{-2016}$
d. $2016!\, x^{-2017}$ Correct Choice
e. $-2015!\, x^{-2016}$

SOLUTION: $f'(x) = 1 + x^{-1}$ $f''(x) = -x^{-2}$ $f'''(x) = 2x^{-3}$ $f^{(4)}(x) = -3 \cdot 2x^{-4}$
$f^{(5)}(x) = 4 \cdot 3 \cdot 2x^{-5}$ $f^{(2017)}(x) = 2016!\, x^{-2017}$

6. Find the line tangent to the curve $y^3 = x^2 - xy$ at $(x,y) = (-2,2)$. Its y-intercept is

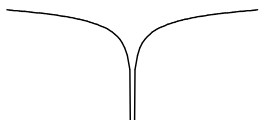
a. $\dfrac{4}{5}$ Correct Choice
b. $\dfrac{3}{5}$
c. $-\dfrac{3}{5}$
d. $-\dfrac{4}{5}$
e. -2

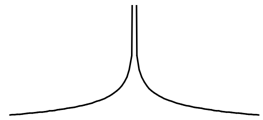
SOLUTION: By implicit differentiation:

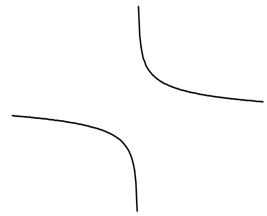
$3y^2 \dfrac{dy}{dx} = 2x - y - x\dfrac{dy}{dx}$ $12\dfrac{dy}{dx} = -4 - 2 + 2\dfrac{dy}{dx}$ $10\dfrac{dy}{dx} = -6$ $\left.\dfrac{dy}{dx}\right|_{(-2,2)} = \dfrac{-3}{5}$

Tan Line: $y = f(-2) + f'(-2)(x+2) = 2 - \dfrac{3}{5}(x+2) = -\dfrac{3}{5}x + \dfrac{4}{5}$ $b = \dfrac{4}{5}$

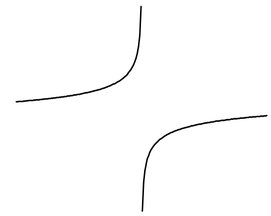
7. Near $x = 2$, the graph of $f(x) = (x-2)^{-1/5}$ has the shape:

a.


b.


c.


Correct Choice

d.


e None of these

SOLUTION: $\lim\limits_{x\to 2^-} (x-2)^{-1/5} = \text{“}\dfrac{1}{(0^-)^{1/5}}\text{”} = -\infty$ $\qquad \lim\limits_{x\to 2^+} (x-2)^{-1/5} = \text{“}\dfrac{1}{(0^+)^{1/5}}\text{”} = \infty$

8. $\lim\limits_{x\to 0} \dfrac{x\sin x}{\cos x - 1} =$

a. -2 Correct Choice
b. -1
c. 0
d. 1
e. 2

SOLUTION: $\lim\limits_{x\to 0} \dfrac{x\sin x}{\cos x - 1} \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{\sin x + x\cos x}{-\sin x} \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{\cos x + \cos x - x\sin x}{-\cos x} = -2$

9. Water is poured onto a tabletop at $\dfrac{dV}{dt} = 3$ cm$^3/\text{sec}$ in such a way that it forms a circular puddle of radius r and constant height of $h = 0.2$ cm. Find the rate that the radius is increasing when it is $r = 5$ cm.
HINT: The puddle is actually a cylinder. What is its volume?

a. $\dfrac{1}{2\pi}$
b. $\dfrac{2\pi}{3}$
c. π
d. $\dfrac{3}{2\pi}$ Correct Choice
e. 2π

SOLUTION: $V = \pi r^2 h = 0.2\pi r^2 \qquad \dfrac{dV}{dt} = 0.4\pi r \dfrac{dr}{dt} \qquad 3 = 0.4\pi 5 \dfrac{dr}{dt} \qquad \dfrac{dr}{dt} = \dfrac{3}{2\pi}$

10. Find the points of inflection of the function $f(x) = \dfrac{1}{20}x^5 - \dfrac{1}{3}x^4$.

a. $x = 0$ and $x = 4$ only
b. $x = 4$ only Correct Choice
c. $x = 0$ only
d. $x = -2$ and $x = 2$ only
e. $x = -4$ and $x = 0$ only

SOLUTION: $f'(x) = \dfrac{1}{4}x^4 - \dfrac{4}{3}x^3 \qquad f''(x) = x^3 - 4x^2 = x^2(x-4) = 0$
Potential inflection points are $x = 0, 4$ but the concavity does not change at $x = 0$.

11. If the initial position and velocity of a car are $x(0) = 3$ and $v(0) = 2$ and its accereration is $a(t) = \sin t + 6t$ find its position at $t = 1$.

a. $7 + \sin 1$
b. $3 + \cos 1$
c. $6 + \sin 1$
d. $7 - \sin 1$ Correct Choice
e. $3 - \cos 1$

SOLUTION: $v(t) = -\cos t + 3t^2 + C \qquad v(0) = -1 + C = 2 \qquad C = 3 \qquad v(t) = -\cos t + 3t^2 + 3$
$x(t) = -\sin t + t^3 + 3t + K \qquad x(0) = K = 3 \qquad x(t) = -\sin t + t^3 + 3t + 3$
$x(1) = -\sin 1 + 1 + 3 + 3 = 7 - \sin 1$

12. If $A(z)$ is the area under the parabola $y = 2 + x^2$ between $x = 1$ and $x = z$ find $A'(3)$.

a. 6
b. 8
c. 9
d. 11 Correct Choice
e. 13

SOLUTION: $A(z) = \int_1^z (2 + x^2)\, dx \qquad A'(z) = \frac{d}{dz} \int_1^z (2 + x^2)\, dx = 2 + z^2$ by the F.T.C.
$A'(3) = 11$

13. Find the area under the parabola $y = 2 + x^2$ between $x = 1$ and $x = 4$.

a. 21
b. 24
c. 27 Correct Choice
d. 30
e. 36

SOLUTION: $A = \int_1^4 (2 + x^2)\, dx = \left[2x + \frac{x^3}{3} \right]_1^4 = \left(8 + \frac{64}{3} \right) - \left(2 + \frac{1}{3} \right) = 6 + \frac{63}{3} = 27$

14. Approximate the area under the parabola $y = 1 + x^2$ between $x = 1$ and $x = 7$ using a Right Riemann Sum with 3 equal intervals and right endpoints.

a. 38
b. 76
c. 86
d. 120
e. 172 Correct Choice

SOLUTION: $a = 1 \qquad b = 7 \qquad n = 3 \qquad \Delta x = \frac{b - a}{n} = \frac{7 - 1}{3} = 2$
$x_i = a + i\Delta x \qquad x_1 = 1 + 2 = 3 \qquad x_2 = 1 + 2 \cdot 2 = 5 \qquad x_3 = 1 + 3 \cdot 2 = 7$
$f(x_1) = 1 + 3^2 = 10 \qquad f(x_2) = 1 + 5^2 = 26 \qquad f(x_3) = 1 + 7^2 = 50$
$$A \approx \sum_{i=1}^{3} f(x_i)\Delta x = 2(f(x_1) + f(x_2) + f(x_3)) = 2(10 + 26 + 50) = 172$$

Part 2 – Work Out Problems (3 questions. Points indicated. No Calculators)

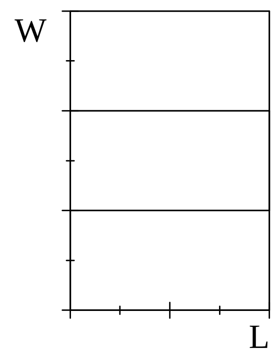
Solve each problem in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

15. (10 points) Let $f(x) = xe^x$ and $g(x) = f^{-1}(x)$ the inverse function of $f(x)$. Find $g'(e)$.

SOLUTION: If $b = g(e)$ then $e = f(b) = be^b$. So $b = 1$.

Further $f'(x) = e^x + xe^x$ and $f'(1) = e^1 + 1e^1 = 2e$. So $g'(e) = \dfrac{1}{f'(1)} = \dfrac{1}{2e}$.

16. (10 points) A rectangular field is surrounded by a fence and divided into 3 pens by 2 additional fences parallel to one side. If the total area is 50 m^2 find the minimum total length of fence.



SOLUTION: $A = LW = 50$ $\quad W = \dfrac{50}{L}$ $\quad F = 4L + 2W = 4L + \dfrac{100}{L}$ $\quad F' = 4 - \dfrac{100}{L^2} = 0$

$4 = \dfrac{100}{L^2}$ $\quad L^2 = 25$ $\quad L = 5$ $\quad W = \dfrac{50}{5} = 10$ $\quad F = 4 \cdot 5 + 2 \cdot 10 = 40$

17. (30 points) Analyze the graph of the function $f(x) = xe^{-2x^2}$. Find each of the following. If none or nowhere or undefined, say so.

a. y-intercepts:

$y = f(0) = 0$ 1

b. x-intercepts:

$xe^{-2x^2} = 0$ when $x = 0$ only 1

c. horizontal asymptote as $x \to +\infty$:

$\lim\limits_{x \to \infty} xe^{-2x^2} = \lim\limits_{x \to \infty} \frac{x}{e^{2x^2}} \overset{\text{l'H}}{=} \lim\limits_{x \to \infty} \frac{1}{e^{2x^2} 4x} = 0$ 1

d. horizontal asymptote as $x \to -\infty$:

$\lim\limits_{x \to -\infty} xe^{-2x^2} = \lim\limits_{x \to -\infty} \frac{x}{e^{2x^2}} \overset{\text{l'H}}{=} \lim\limits_{x \to -\infty} \frac{1}{e^{2x^2} 4x} = 0$ 1

e. vertical asymptotes:

none 1

f. first derivative:

$f'(x) = e^{-2x^2} + xe^{-2x^2}(-4x) = e^{-2x^2}(1 - 4x^2)$ 1

g. critical points:

$f'(x) = 0 \qquad 1 - 4x^2 = 0 \qquad x = \pm\frac{1}{2}$ 1

h. intervals where increasing:

Put the critical points $x = \pm\frac{1}{2}$ on a number line and test the 3 intervals:

$f'(-1) = -3e^{-2} < 0 \quad f'(0) = 1 > 0 \quad f'(1) = -3e^{-2} < 0$

increasing on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ 3

i. intervals where decreasing:

decreasing on $\left(-\infty, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$ 2

j. local maxima $\left(x \text{ and } y \text{ coordinates}\right)$:

$f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1/2}$

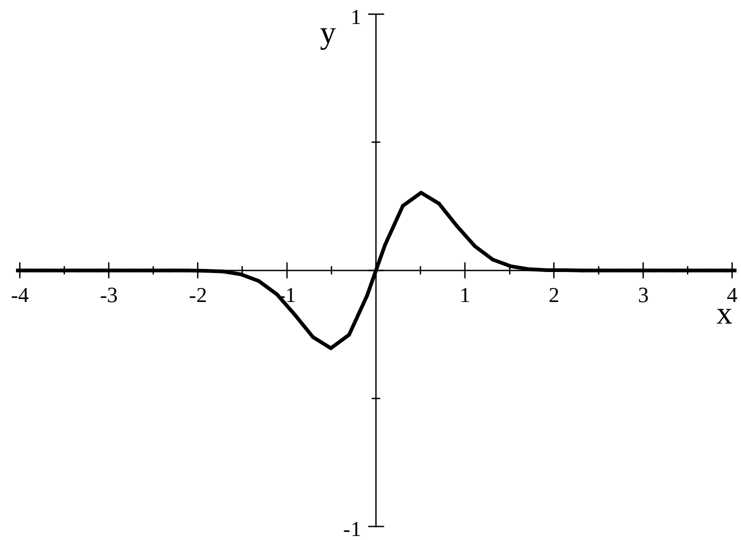
local maximum at $\left(\frac{1}{2}, \frac{1}{2}e^{-1/2}\right)$ 2

k. local minima $\left(x \text{ and } y \text{ coordinates}\right)$:

$f\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-1/2}$

local minimum at $\left(-\frac{1}{2}, -\frac{1}{2}e^{-1/2}\right)$ 2

l. Roughly graph the function: 5



m. second derivative:

$$f''(x) = -8xe^{-2x^2} + (1-4x^2)e^{-2x^2}(-4x) = -4xe^{-2x^2}(3-4x^2)$$ 2

n. inflection points $(x$ coordinate only$)$:

$$x = 0, \sqrt{\frac{3}{4}}, -\sqrt{\frac{3}{4}}$$ 2

o. intervals where concave up:

Put the inflection points $x = 0, \pm\sqrt{\frac{3}{4}}$ on a number line and test the 4 intervals:

$f''(-1) = -4e^{-2} < 0$ $f''\left(-\frac{1}{2}\right) = 4e^{-1/2} > 0$ $f''\left(\frac{1}{2}\right) = -4e^{-1/2} < 0$ $f''(1) = 4e^{-2} > 0$

concave up on $\left(-\sqrt{\frac{3}{4}}, 0\right)$ and $\left(\sqrt{\frac{3}{4}}, \infty\right)$ 3

p. intervals where concave down:

concave down on $\left(-\infty, -\sqrt{\frac{3}{4}}\right)$ and $\left(0, \sqrt{\frac{3}{4}}\right)$ 2