MATH 152
Exam1
Spring 2000
Test Form A

Solutions

Part I is multiple choice. There is no partial credit.

Part II is work out. Show all your work. Partial credit will be given.

You may not use a calculator.

Formulas:

$\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$

$\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$

$S_n = \frac{1}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]\Delta x$

$|E_T| \le \frac{K(b-a)^3}{12n^2}$ where $K \ge |f''(x)|$ for $a \le x \le b$

$|E_S| \le \frac{K(b-a)^5}{180n^4}$ where $K \ge \left|f^{(4)}(x)\right|$ for $a \le x \le b$

$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$

$\int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$

$\int \ln x\, dx = x\ln x - x + C$

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. Compute $\int_0^{1/2} xe^{2x}\,dx$.

a. $\frac{1}{2}e - \frac{1}{4}$

b. $\frac{1}{4}e^2$

c. $\frac{1}{4}(e^2 - 1)$

d. $\frac{1}{4}$ correctchoice

e. $\frac{3}{4}$

Integrate by parts with $\begin{array}{ll} u = x & dv = e^{2x}dx \\ du = dx & v = \frac{1}{2}e^{2x} \end{array}$ to get

$$\int_0^{1/2} xe^{2x}\,dx = \left[\frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]_0^{1/2} = \left[\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right]_0^{1/2} = \left[\frac{1}{4}e^1 - \frac{1}{4}e^1\right] - \left[\frac{0}{2}e^0 - \frac{1}{4}e^0\right] = \frac{1}{4}$$

2. Compute $\int_0^{\pi} \cos^2(x)\,dx$.

a. 1

b. $\frac{\pi}{2}$ correctchoice

c. 2

d. $\frac{\pi}{2} - 1$

e. π

$$\int_0^{\pi} \cos^2(x)\,dx = \int_0^{\pi} \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}\left[\pi + \frac{\sin 2\pi}{2}\right] - \frac{1}{2}\left[0 + \frac{\sin 0}{2}\right] = \frac{\pi}{2}$$

3. Find the average value of the function $y = xe^{x^2}$ on the interval $0 \le x \le 2$.

a. $\frac{1}{2}(e^4 - 1)$

b. $\frac{1}{4}(e^4 - 1)$ correctchoice

c. e

d. $\frac{1}{2}e^4$

e. e^4

Use the substitution $u = x^2$. So $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$:

$$y_{\text{ave}} = \frac{1}{2}\int_0^2 xe^{x^2}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}e^4 - \frac{1}{4}e^0 = \frac{1}{4}e^4 - \frac{1}{4}$$

4. Use the Trapezoid Rule with $n = 2$ to approximate $\int_1^5 \frac{1}{x}\,dx$.

a. $\frac{14}{15}$

b. $\frac{23}{15}$

c. $\frac{28}{15}$ correctchoice

d. $\frac{46}{15}$

e. $\frac{76}{45}$

$$\Delta x = \frac{5-1}{2} = 2 \qquad f(x) = \frac{1}{x}$$

$$T_2 = \left[\frac{1}{2}f(1) + f(3) + \frac{1}{2}f(5)\right]\Delta x = \left[\frac{1}{2}1 + \frac{1}{3} + \frac{1}{2}\bullet\frac{1}{5}\right]2 = 2\left[\frac{15+10+3}{30}\right] = \frac{28}{15}$$

5. If you use the Trapezoid Rule with $n = 8$ to estimate $\int_1^5 \frac{1}{x}\,dx$ you obtain the approximation $T_8 = 1.628968$. (Don't compute this.) Use the Trapezoid Error formula

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{where } K \ge |f''(x)| \text{ for } a \le x \le b$$

to find an upper bound for the error in this approximation.

a. $|E_T| \le \frac{1}{6}$ correctchoice

b. $|E_T| \le \frac{1}{60}$

c. $|E_T| \le \frac{1}{72}$

d. $|E_T| \le \frac{1}{300}$

e. $|E_T| \le \frac{1}{750}$

$$f(x) = \frac{1}{x} \qquad f'(x) = \frac{-1}{x^2} \qquad f''(x) = \frac{2}{x^3} \qquad \text{Max at } x = 1: \qquad K = f''(1) = \frac{2}{1} = 2$$

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{2(5-1)^3}{12(8)^2} = \frac{2\bullet 64}{12\bullet 64} = \frac{1}{6}$$

6. Compute $\int_0^{\pi/4} \tan^2(x)\sec^4(x)\,dx$

a. $\frac{\sqrt{2}}{2} - \ln\left(1 - \frac{\sqrt{2}}{2}\right)$

b. $\frac{\sqrt{2}}{2} + \ln\left(1 - \frac{\sqrt{2}}{2}\right)$

c. $\frac{8}{15}$ correctchoice

d. $\frac{4}{35}$

e. 4

Let $u = \tan x$. Then $du = \sec^2 x\,dx$ and $\sec^2 x = \tan^2 x + 1 = u^2 + 1$. So

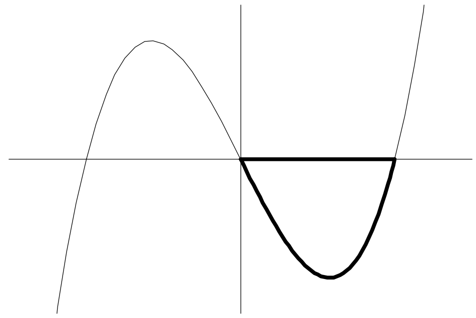
$$\int_0^{\pi/4} \tan^2(x)\sec^4(x)\,dx = \int_0^{\pi/4} \tan^2(x)\sec^2(x)\sec^2(x)\,dx = \int u^2(u^2+1)\,du = \int (u^4 + u^2)\,du$$

$$= \left[\frac{u^5}{5} + \frac{u^3}{3}\right] = \left[\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3}\right]_0^{\pi/4} = \left[\frac{\tan^5\left(\frac{\pi}{4}\right)}{5} + \frac{\tan^3\left(\frac{\pi}{4}\right)}{3}\right] - 0 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$y = x^3 - x$ and $y = 0$ for $x \geq 0$

about the y-axis.



a. $\frac{4}{15}\pi$ correctchoice

b. $\frac{1}{2}\pi$

c. $\frac{2}{15}\pi$

d. $\frac{8}{105}\pi$

e. $\frac{16}{105}\pi$

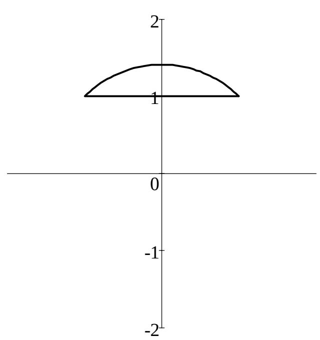
Use an x-integral and cylinders. The radius is $r = x$. The height is $h = 0 - (x^3 - x) = x - x^3$.

$$V = \int_0^1 2\pi rh\,dx = \int_0^1 2\pi x(x - x^3)\,dx = 2\pi\int_0^1 (x^2 - x^4)\,dx = 2\pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 2\pi\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{4}{15}\pi$$

8. Which integral gives the volume of the solid obtained by rotating the region bounded by

$x^2 + y^2 = 2$ and $y = 1$

about the x-axis?



a. $\int_{-1}^{1} \pi\left((2 - x^2)^2 - 1\right) dx$

b. $\int_{-\sqrt{2}}^{\sqrt{2}} \pi\left(\sqrt{2 - x^2} - 1\right)^2 dx$

c. $\int_0^1 2\pi x(1 - x^2)\,dx$

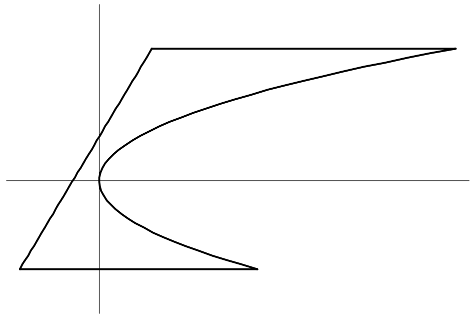
d. $\int_0^1 2\pi x\left(\sqrt{2 - x^2} - 1\right) dx$

e. $\int_{-1}^{1} \pi(1 - x^2)\,dx$ correctchoice

Use an x-integral and washers. The outer radius is $R = \sqrt{2 - x^2}$. The inner radius is $r = 1$.
The two curves intersect when $x^2 + 1^2 = 2$ or $x = \pm 1$. So these are the limits.

$$V = \int_{-1}^{1} \pi(R^2 - r^2)\,dx = \int_{-1}^{1} \pi\left(\sqrt{2 - x^2}^{\,2} - 1^2\right) dx = \int_{-1}^{1} \pi(2 - x^2 - 1)\,dx = \int_{-1}^{1} \pi(1 - x^2)\,dx$$

9. Find the area of the region bounded by
$x = 3y^2$, $x - 2y = -2$,
$y = -2$ and $y = 3$.



a. 8
b. 24
c. 40 correctchoice
d. 56
e. 64

Use a y-integral.

$$A = \int_{-2}^{3} (\text{right}) - (\text{left})\,dy = \int_{-2}^{3} (3y^2) - (2y - 2)\,dy = \int_{-2}^{3} (3y^2 - 2y + 2)\,dy$$

$$= [y^3 - y^2 + 2y]_{-2}^{3} = [27 - 9 + 6] - [-8 - 4 - 4] = 40$$

10. Which of the following definite integrals equals $\int_1^3 \sqrt{x^2 - 2x + 5}\,dx$?

a. $4\int_0^{\pi/2} \sin\theta\,d\theta$
b. $2\int_0^{\pi/2} \sin^2\theta\,d\theta$
c. $2\int_0^{\pi/4} \tan\theta\,d\theta$
d. $4\int_0^{\pi/4} \sec\theta\,d\theta$
e. $4\int_0^{\pi/4} \sec^3\theta\,d\theta$ correctchoice

Complete the square:

$$\int_1^3 \sqrt{x^2 - 2x + 5}\,dx = \int_1^3 \sqrt{(x-1)^2 + 4}\,dx$$

Use the trig substitution: $x - 1 = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$. So

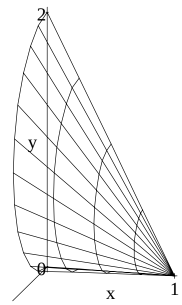
$$\int_1^3 \sqrt{x^2 - 2x + 5}\,dx = \int_1^3 \sqrt{4\tan^2\theta + 4}\;2\sec^2\theta\,d\theta = 4\int_1^3 \sqrt{\tan^2\theta + 1}\;\sec^2\theta\,d\theta = 4\int_0^{\pi/4} \sec^3\theta\,d\theta$$

Part II: Work Out (10 points each)

Show all your work. Partial credit will be given.

You may not use a calculator.

11. An object has a base that is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$. The crosssections perpendicular to the x-axis are semicircles. Find the volume of the object. You must use an integral.



The upper edge of the triangle is $y = 2 - 2x$. This is the diameter of the semicircles. So the radius is $r = 1 - x$ and the area is

$$A(x) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1-x)^2 = \frac{1}{2}\pi(x-1)^2$$

So the volume is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \frac{1}{2}\pi(x-1)^2\,dx = \frac{1}{2}\pi\frac{(x-1)^3}{3}\Big|_0^1 = 0 - \pi\frac{(-1)^3}{6} = \frac{\pi}{6}$$

12. Compute $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$.

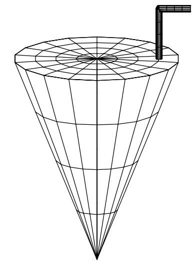
Use the trig substitution $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$ and

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \int \frac{1}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\,2\sec\theta\tan\theta\,d\theta = \frac{1}{4}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + C = \frac{1}{4}\frac{\sqrt{x^2-4}}{x} + C$$

where we have used a triangle whose hypotenuse is x, adjacent side is 2 and opposite side is $\sqrt{x^2-4}$.

13. A tank has the shape of a circular cone with height 20 m, radius 8 m and vertex down. It has a spout which extends 5 m above the top of the tank. The tank is initially full of water. Set up the integral that gives the work required to pump all the water out of the spout. Do not evaluate the integral.



water density $= 1000 \frac{\text{kg}}{\text{m}^3}$ $\quad g = 9.8 \frac{\text{m}}{\text{sec}^2}$

We choose to measure y up from the vertex of the cone. The slice at height y with thickness dy is a circle of radius r. So its volume is $dV = \pi r^2\, dy$. The radius r satisfies the similar triangle relation $\frac{r}{y} = \frac{8}{20}$. So $r = \frac{2}{5} y$ and the volume is $dV = \pi \frac{4}{25} y^2\, dy$. Hence the weight (force) is $dF = \rho g\, dV = \rho g \pi \frac{4}{25} y^2\, dy$. This slice at height y must be lifted to 25 m. So it is lifted a distance $D = 25 - y$. So the work is

$$W = \int_0^{20} D\, dF == \int_0^{20} (25 - y)\, \rho g \pi \frac{4}{25} y^2\, dy = \frac{4}{25} 1000 \bullet 9.8 \pi \int_0^{20} (25 - y)\, y^2\, dy$$

Other answers are possible if you put the origin at different heights.

14. Compute $\int_1^e x \ln x\, dx$.

Integrate by parts with $\quad u = \ln x \quad dv = x\, dx \quad du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$ $\quad$ Then

$$\int_1^e x \ln x\, dx = \left[\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x}\, dx \right]_1^e = \left[\frac{x^2}{2} \ln x - \int \frac{x}{2}\, dx \right]_1^e = \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_1^e$$

$$= \left[\frac{e^2}{2} \ln e - \frac{e^2}{4} \right] - \left[\frac{1}{2} \ln 1 - \frac{1}{4} \right] = \left[\frac{e^2}{2} - \frac{e^2}{4} \right] - \left[-\frac{1}{4} \right] = \frac{e^2}{4} + \frac{1}{4}$$

15.

a. Find the partial fraction expansion for $\frac{x-2}{x^3+x}$.

Factor the denominator and write the general expansion:

$$\frac{x-2}{x^3+x} = \frac{x-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Clear the denominator:

$$x-2 = A(x^2+1) + (Bx+C)(x)$$

Plug in numbers and solve for A, B and C:

$x = 0$: $\Rightarrow$ $-2 = A(1)$ $\Rightarrow$ $A = -2$

$x = 1$: $\Rightarrow$ $-1 = A(2) + (B+C)(1) = -4 + B + C$ $\Rightarrow$ $B + C = 3$

$x = -1$: $\Rightarrow$ $-3 = A(2) + (-B+C)(-1) = -4 + B - C$ $\Rightarrow$ $B - C = 1$

So:

$$A = -2 \qquad B = 2 \qquad C = 1$$

and

$$\frac{x-2}{x^3+x} = \frac{-2}{x} + \frac{2x+1}{x^2+1}$$

b. Then compute $\int \frac{x-2}{x^3+x}\,dx$.

$$\int \frac{x-2}{x^3+x}\,dx = \int \frac{-2}{x} + \frac{2x+1}{x^2+1}\,dx = \int \frac{-2}{x}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= -2\ln|x| + \ln|x^2+1| + \arctan x + C$$