MATH 152
Exam 2
Spring 2000
Test Form A
- Solutions -

Part I is multiple choice. There is no partial credit.

Part II is work out. Show all your work. Partial credit will be given.

You may not use a calculator.

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. Compute $\int_{-1}^{1} \frac{1}{x^6}\,dx$.

a. 0
b. $\frac{2}{5}$
c. $-\frac{2}{7}$
d. $-\frac{2}{5}$
e. Divergent correctchoice

$\int_{-1}^{1} \frac{1}{x^6}\,dx = \int_{-1}^{0} \frac{1}{x^6}\,dx + \int_{0}^{1} \frac{1}{x^6}\,dx$ and $\int_{0}^{1} \frac{1}{x^6}\,dx = \lim_{a\to 0}\left[\frac{x^{-5}}{-5}\right]_a^1 = \left[\frac{-1}{5}\right] - \lim_{a\to 0^+}\left[\frac{-1}{5a^5}\right] = \infty$

Since $\int_{0}^{1} \frac{1}{x^6}\,dx$ is divergent, so is $\int_{-1}^{1} \frac{1}{x^6}\,dx$.

2. Find the solution to $\frac{dy}{dx} = -\frac{x}{y}$ satisfying the initial condition $y(3) = 4$. Then $y(0) =$

a. 0
b. 1
c. 3
d. 5 correctchoice
e. 6

Separable differential equation. Separate the variables and integrate:

$$y\,dy = -x\,dx \quad\Rightarrow\quad \int y\,dy = -\int x\,dx \quad\Rightarrow\quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

Use the initial condition $x = 3$ $y = 4$:

$$\frac{4^2}{2} = -\frac{3^2}{2} + C \quad\Rightarrow\quad C = \frac{16}{2} + \frac{9}{2} = \frac{25}{2}$$

The solution is:

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{25}{2} \quad\Rightarrow\quad x^2 + y^2 = 25 \quad\Rightarrow\quad y = \sqrt{25 - x^2}$$

So $y(0) = 5$.

3. A tank contains 500 liters of water with 10 kg of sugar dissolved. Sugar water that contains $\frac{1}{10}$ kg of sugar per liter of water flows into the tank at the rate of 7 liters per minute. Sugar water that contains $\frac{1}{20}$ kg of sugar per liter of water flows into the tank at the rate of 8 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the rate of 15 liters per minute. Write down the initial value problem for $S(t)$, the amount of sugar in the tank at time t.

a. $\frac{dS}{dt} = 10t + \frac{3}{10} - \frac{S}{500}$ with $S(0) = 100$

b. $\frac{dS}{dt} = \frac{11}{10} - \frac{3S}{100}$ with $S(0) = 10$ correctchoice

c. $\frac{dS}{dt} = \frac{3}{20} - \frac{S}{500}$ with $S(0) = 10$

d. $\frac{dS}{dt} = \frac{3}{20} - \frac{3S}{100}$ with $S(0) = \frac{1}{50}$

e. $\frac{dS}{dt} = \frac{11}{10}t - \frac{S}{500}$ with $S(0) = 100$

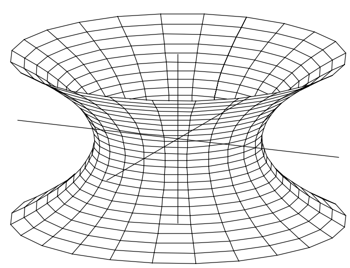
$$\frac{dS}{dt} = \underbrace{\frac{1}{10}\frac{\text{kg}}{\text{L}} \bullet \frac{7\text{ L}}{\text{min}} + \frac{1}{20}\frac{\text{kg}}{\text{L}} \bullet \frac{8\text{ L}}{\text{min}}}_{\text{in}} - \underbrace{\frac{S\text{ kg}}{500\text{ L}} \bullet \frac{15\text{ L}}{\text{min}}}_{\text{out}} = \frac{11}{10} - \frac{3S}{100}$$

$S(0) = 10$ kg

4. Which integral gives the surface area of the "spool" obtained by revolving the curve

$y^2 - x + 1 = 0$ for $-1 \le y \le 1$

about the y-axis.



a. $\int_{-1}^{1} 2\pi(1 + y^2)\sqrt{1 + 4y^2}\, dy$ correctchoice

b. $\int_{-1}^{1} 2\pi y\sqrt{1 + 4y^2}\, dy$

c. $\int_{-1}^{1} \pi y\sqrt{1 + 2y}\, dy$

d. $\int_{-1}^{1} 2\pi(1 + 4y^2)\sqrt{1 + y^2}\, dy$

e. $\int_{-1}^{1} 2\pi(1 + y^2)\sqrt{1 + 2y^2}\, dy$

y-integral, $r = x = y^2 + 1$, $\frac{dx}{dy} = 2y$

$$A = \int 2\pi r\, ds = \int_{-1}^{1} 2\pi(y^2 + 1)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{-1}^{1} 2\pi(y^2 + 1)\sqrt{1 + 4y^2}\, dy$$

5. Find the x-coordinate of the centroid (center of mass, $\bar{x}$) of the region between $y = x^3$ and the x-axis for $0 \le x \le 2$.

a. $\frac{4}{5}$

b. 4

c. $\frac{8}{5}$ correctchoice

d. $\frac{32}{5}$

e. $\frac{2}{3}$

The area is $A = \int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.$

The first moment is $1^{st}\text{-mom} = \int_0^2 x^4\,dx = \left[\frac{x^5}{5}\right]_0^2 = \frac{32}{5}.$

The x-coordinate of the centroid is $\bar{x} = \frac{1^{st}\text{-mom}}{A} = \frac{32}{5 \bullet 4} = \frac{8}{5}$

6. Find the limit of the sequence $a_n = \frac{\ln(n^2+1)}{\ln(n)}.$

a. 0

b. $\ln 2$

c. 2 correctchoice

d. e

e. ∞

$$\lim_{n\to\infty} \frac{\ln(n^2+1)}{\ln(n)} \overset{l'H}{=} \lim_{n\to\infty} \frac{\frac{2n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2}{n^2+1} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{n^2}} = 2$$

7. Which of the following series are convergent?

(i) $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ (ii) $\sum_{n=1}^{\infty} \frac{2^n}{n+3^n}$

a. both (i) and (ii) correctchoice

b. (i) only

c. (ii) only

d. neither

(i) converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{100^{n+1}}{(n+1)!} \frac{n!}{100^n} = \lim_{n\to\infty} \frac{100}{(n+1)} = 0 < 1.$

(ii) converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{n+1+3^{n+1}} \frac{n+3^n}{2^n} = \frac{2}{3} < 1$

or by comparison to $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ which is a geometric series with $r = \frac{2}{3} < 1.$

8. Which of the following series are convergent?

(i) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ (ii) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

a. both (i) and (ii)
b. (i) only
c. (ii) only
d. neither correctchoice

(i) diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a p-series with $p = \frac{1}{2} < 1$

since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

or by comparison to $\sum_{n=1}^{\infty} \frac{1}{n+1}$ which is a harmonic series since $\frac{\sqrt{n}}{n+1} \geq \frac{1}{n+1}$.

(ii) diverges by the n^{th}-Term Divergence Test: $\lim_{n\to\infty} \frac{n}{\ln n} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty \neq 0.$

9. Compute $\sum_{n=0}^{\infty} \frac{5^{n+1}}{4^n}$.

a. -20
b. $\frac{20}{9}$
c. 20
d. 25
e. divergent correctchoice

This is a geometric series whose ratio is $r = \frac{5}{4} > 1$. So the series diverges.

10. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

a. is absolutely convergent.
b. is convergent but not absolutely convergent. correctchoice
c. is divergent to $+\infty$.
d. is divergent to $-\infty$.
e. is divergent but not to $\pm\infty$.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test since $\frac{1}{\sqrt{n}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

The related absolute series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series since $p = \frac{1}{2} < 1$.

So the original series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is not absolutely convergent.

Part II: Work Out (10 points each)

Show all your work. Partial credit will be given.

You may not use a calculator.

11. Find the solution of $\dfrac{dy}{dx} - 3y = 6e^{-x}$ satisfying $y(0) = 5$. Solve for y.

$$I = e^{\int(-3)\,dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}e^{-x} \quad\Rightarrow\quad \frac{d}{dx}(e^{-3x}y) = 6e^{-4x} \quad\Rightarrow\quad e^{-3x}y = \int 6e^{-4x}\,dx = \frac{6e^{-4x}}{-4} + C$$

$$e^{-3x}y = -\frac{3}{2}e^{-4x} + C \quad\Rightarrow\quad y = -\frac{3}{2}e^{-x} + Ce^{3x}$$

$$x = 0, \quad y = 5 \quad\Rightarrow\quad e^{0}5 = -\frac{3}{2}e^{0} + C \quad\Rightarrow\quad C = \frac{13}{2}$$

$$e^{-3x}y = -\frac{3}{2}e^{-4x} + \frac{13}{2} \quad\Rightarrow\quad y = -\frac{3}{2}e^{-x} + \frac{13}{2}e^{3x}$$

12. Consider the series $S = \sum_{n=1}^{\infty}\left(\dfrac{n}{n+1} - \dfrac{n+1}{n+2}\right)$

a. Write out the third partial sum $S_3 = \sum_{n=1}^{3}\left(\dfrac{n}{n+1} - \dfrac{n+1}{n+2}\right)$ and add it up.

$$S_3 = \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{4}{5}\right) = \frac{1}{2} - \frac{4}{5} = \frac{5-8}{10} = -\frac{3}{10}$$

b. Compute the sum of the infinite series $S = \sum_{n=1}^{\infty}\left(\dfrac{n}{n+1} - \dfrac{n+1}{n+2}\right)$.

$$S_k = \sum_{n=1}^{k}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right) = \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \cdots + \left(\frac{k}{k+1} - \frac{k+1}{k+2}\right) = \frac{1}{2} - \frac{k+1}{k+2}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{k+1}{k+2}\right) = \frac{1}{2} - 1 = -\frac{1}{2}$$

13. Find the arc length of the parametric curve $x = t^3$, $y = 3t^2$ between $t = 0$ and $t = \sqrt{12}$.

t-integral $\quad \dfrac{dx}{dt} = 3t^2 \qquad \dfrac{dy}{dt} = 6t$

$$L = \int_0^{\sqrt{12}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\sqrt{12}} \sqrt{(3t^2)^2 + (6t)^2}\,dt = \int_0^{\sqrt{12}} \sqrt{9t^4 + 36t^2}\,dt = \int_0^{\sqrt{12}} 3t\sqrt{t^2+4}\,dt$$

Let $u = t^2 + 4$ Then $du = 2t\,dt$ or $\frac{1}{2}du = t\,dt$

$$L = \frac{3}{2}\int \sqrt{u}\,du = u^{3/2} = (t^2+4)^{3/2}\Big|_0^{\sqrt{12}} = 16^{3/2} - 4^{3/2} = 4^3 - 2^3 = 64 - 8 = 56$$

14. Consider the series $S = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

a. Prove the series $S = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges. Be sure to name the convergence test you use.

We apply the Integral Test: Let $u = \ln n$. Then $du = \frac{1}{n}\,dn$ So

$$\int_2^{\infty} \frac{1}{n(\ln n)^2}\,dn = \int_{n=2}^{\infty} \frac{1}{u^2}\,du = \frac{-1}{u} = \left[\frac{-1}{\ln n}\right]_2^{\infty} = 0 - \frac{-1}{\ln 2} = \frac{1}{\ln 2}$$

Since the integral converges, the series converges.

b. If you approximate the series S by the partial sum $S_9 = \sum_{n=2}^{9} \frac{1}{n(\ln n)^2}$, find an upper bound for the error $|R_9| = |S - S_9|$ in the approximation and justify your estimate.

The remainder is

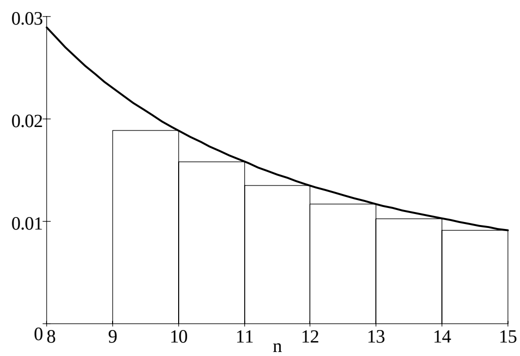
$$R_9 = S - S_9 = \sum_{n=10}^{\infty} \frac{1}{n(\ln n)^2}$$

The function $\frac{1}{n(\ln n)^2}$ is decreasing.

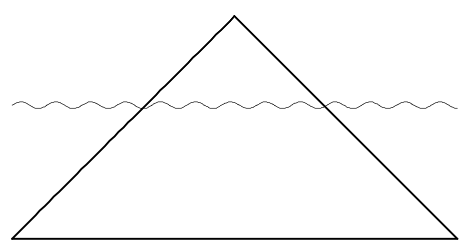
The remainder is bounded by

$$R_9 < \int_9^{\infty} \frac{1}{n(\ln n)^2}\,dn = \left[\frac{-1}{\ln n}\right]_9^{\infty}$$

$$= 0 - \frac{-1}{\ln 9} = \frac{1}{\ln 9}$$



15. A triangular plate is suspended vertically in water as shown. The height of the triangle is 5 ft. Its base is 10 ft, and it is submerged 3 ft into the water. Find the force on one face of the plate.
($\rho g = 62.5 \frac{\text{lb}}{\text{ft}^3}$)



Measure y down from the tip of the triangle. So the water is at $2 \le y \le 5$. Let w be the width of the triangle at position y. So by similar triangles

$$\frac{w}{y} = \frac{10}{5} = 2 \qquad \Rightarrow \qquad w = 2y$$

Consequently, the area of a slice of height dy is

$$dA = w\,dy = 2y\,dy$$

The height of the water above this slice is $h = y - 2$. So the force on the plate is

$$F = \int_2^5 \rho g h\,dA = \rho g \int_2^5 (y-2)2y\,dy = \rho g \int_2^5 (2y^2 - 4y)\,dy = \rho g \left[2\frac{y^3}{3} - 2y^2\right]_2^5$$

$$= \rho g\left(\frac{250}{3} - 50\right) - \rho g\left(\frac{16}{3} - 8\right) = 36\rho g = 2250$$