Student (Print) ______________________________ Section __________

Last, First Middle

Student (Sign) ______________________________

Student ID ______________________________

Instructor ______________________________

# MATH 152

# Exam 3

# Spring 2000

# Test Form A

# Solutions

| | |
|---|---|
| 1-10 | /50 |
| 11 | /10 |
| 12 | /15 |
| 13 | /15 |
| 14 | /10 |
| TOTAL | |

Part I is multiple choice. There is no partial credit.

Part II is work out. Show all your work. Partial credit will be given.

You may not use a calculator.

## Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

**1.** Find the values of $x$ such that the vectors $\langle x,-1,3\rangle$ and $\langle 2,-5,x\rangle$ are orthogonal.

- **a.** $-1$ only   correctchoice
- **b.** $0$ only
- **c.** $1$ only
- **d.** $0$ and $1$ only
- **e.** $1$ and $-1$ only

$\langle x,-1,3\rangle \bullet \langle 2,-5,x\rangle = 2x+5+3x = 5+5x = 0$ at $x=-1$

**2.** Compute $\lim\limits_{x\to 0} \dfrac{e^{x^3}-1-x^3}{x^6}$

HINT: The series for $e^x$ may be helpful.

- **a.** $0$
- **b.** $\dfrac{1}{3!}$
- **c.** $-\dfrac{1}{3!}$
- **d.** $\dfrac{1}{2}$   correctchoice
- **e.** $\infty$

$e^x = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots$ ........................................................................ Known Taylor Series

$e^{x^3} = 1+x^3+\dfrac{x^6}{2!}+\dfrac{x^9}{3!}+\cdots$ ........................................................................Substitute $x\to x^3$

$$\lim_{x\to 0}\frac{e^{x^3}-1-x^3}{x^6} = \lim_{x\to 0}\frac{\left(1+x^3+\frac{x^6}{2!}+\frac{x^9}{3!}+\cdots\right)-1-x^3}{x^6} = \frac{1}{2}$$

**3.** Consider the parametric curve $\vec{r}(t) = \langle t,\sin t,t^3\rangle$. Find parametric equations for the line tangent to the curve at $t=\pi$.

- **a.** $x = 1+\pi t,\quad y=-t,\quad z=3\pi^2+\pi^3 t$
- **b.** $x = 1+\pi t,\quad y=-1,\quad z=3\pi^2+\pi^3 t$
- **c.** $x = \pi+t,\quad y=-1+t\cos t,\quad z=\pi^3+3t^3$
- **d.** $x = \pi+t,\quad y=t\cos t,\quad z=\pi^3+3t^3$
- **e.** $x = \pi+t,\quad y=-t,\quad z=\pi^3+3\pi^2 t$   correctchoice

$\vec{r}(t) = \langle t,\sin t,t^3\rangle \qquad \vec{r}'(t) = \langle 1,\cos t,3t^2\rangle \qquad \vec{r}(\pi) = \langle \pi,0,\pi^3\rangle \qquad \vec{r}'(\pi) = \langle 1,-1,3\pi^2\rangle$

Tangent line is

$(x,y,z) = \vec{r}(\pi)+t\vec{r}'(\pi) = \langle \pi,0,\pi^3\rangle + t\langle 1,-1,3\pi^2\rangle = \langle \pi+t,-t,\pi^3+3\pi^2 t\rangle$

**4**. Find the Taylor series for $f(x) = x^2 + 3$ about $x = 2$.

**a**. $7 + 4(x-2) + (x-2)^2$ correctchoice

**b**. $7 + 4(x-2) + 2(x-2)^2 + 4(x-2)^3$

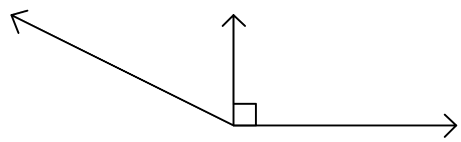
**c**. $7 + 4(x-2) + (x-2)^2 + \frac{2}{3}(x-2)^4 + \cdots$

**d**. $7 + 4(x-2) + 2(x-2)^2 + 4(x-2)^3 + 2(x-2)^4 + \cdots$

**e**. $7 + 4(x-2) + 2(x-2)^2 + \frac{2}{3}(x-2)^3 + \frac{4}{3}(x-2)^4$

$f(x) = x^2 + 3$ is a quadratic polynomial. So the Taylor series cannot be higher than $2^{\text{nd}}$ degree. The answer must be $7 + 4(x-2) + (x-2)^2$. If you do compute it, $f(2) = 7$, $f'(2) = 4$ and $f''(2) = 2$. All higher derivatives are zero.

**5**. The vectors $\vec{a}$, $\vec{b}$ and $\vec{c} = \vec{b} - \vec{a}$ all lie in the **same plane** as shown in the diagram. Which of the following statements is TRUE?



**a**. $\vec{a} \times \vec{b} = \vec{0}$.

**b**. $\vec{a} \times \vec{b}$ points into the page.

**c**. $(\vec{a} \times \vec{b}) \bullet \vec{c} = 0$. correctchoice

**d**. $\vec{b} \times (\vec{a} \times \vec{c})$ points in the direction of $-\vec{a}$.

**e**. None of These

$\vec{a} \times \vec{b} \neq \vec{0}$ because they are not parallel.
$\vec{a} \times \vec{b}$ points out of the page by the right hand rule, not in.
$(\vec{a} \times \vec{b}) \bullet \vec{c} = 0$ because $(\vec{a} \times \vec{b}) \perp \vec{c}$. TRUE
$\vec{b} \times (\vec{a} \times \vec{c})$ points to the right by the right hand rule, not left.

**6.** Find a power series centered at $x = 0$ for the function $f(x) = \frac{x}{1-8x^3}$, and determine its radius of convergence.

**a.** $\sum_{n=0}^{\infty}(-1)^n 8^n x^{3n+1}$ $\quad R = \frac{1}{8}$

**b.** $\sum_{n=0}^{\infty}(-1)^n 8^n x^{3n+1}$ $\quad R = 8$

**c.** $\sum_{n=0}^{\infty}\frac{8^n}{n!}x^{3n+1}$ $\quad R = 2$

**d.** $\sum_{n=0}^{\infty}8^n x^{3n+1}$ $\quad R = \frac{1}{8}$

**e.** $\sum_{n=0}^{\infty}8^n x^{3n+1}$ $\quad R = \frac{1}{2}$ $\quad$ correctchoice

$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$ for $|x| < 1$ ........................................ Geometric Series

$\frac{1}{1-8x^3} = \sum_{n=0}^{\infty}(8x^3)^n = \sum_{n=0}^{\infty}8^n x^{3n}$ for $|8x^3| < 1$ ........................................ Substitute $x \to 8x^3$

$\frac{x}{1-8x^3} = \sum_{n=0}^{\infty}8^n x^{3n+1}$ for $|8x^3| < 1$ ........................................ Multiply by $x$

$|8x^3| < 1 \quad \Rightarrow \quad |x^3| < \frac{1}{8} \quad \Rightarrow \quad |x| < \frac{1}{2} \quad \Rightarrow \quad R = \frac{1}{2}$

**7.** Find the distance from the point $(3,-2,4)$ to the center of the sphere $(x-1)^2 + (y+1)^2 + (z-2)^2 = 4$

**a.** 2

**b.** 3 $\quad$ correctchoice

**c.** 9

**d.** $\sqrt{61}$

**e.** 61

The center of the sphere is $(1,-1,2)$. So the distance is

$$\sqrt{(3-1)^2 + (-2--1)^2 + (4-2)^2} = \sqrt{4+1+4} = 3$$

**8**. Let $f(x) = \sin(x^2)$. Compute $f^{(14)}(0)$, the $14^{\text{th}}$ derivative of $f(x)$ evaluated at $0$.
HINT: Use a series for $\sin(x^2)$.

**a**. $-\dfrac{1}{14! \bullet 7!}$

**b**. $\dfrac{7!}{14!}$

**c**. $\dfrac{14!}{7!}$

**d**. $-\dfrac{14!}{7!}$ correctchoice

**e**. $-14! \bullet 7!$

$\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ ..........Known Taylor Series

$\sin(x^2) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$ ..........Substitute $x \to x^2$

$f(x) = f(0) + f'(0)x + \dfrac{1}{2}f''(0)x^2 + \cdots + \dfrac{1}{14!}f^{(14)}(0)x^{14} + \cdots$ .......... Standard Taylor Series

Equate coefficients of $x^{14}$: $\dfrac{1}{14!}f^{(14)}(0) = -\dfrac{1}{7!} \quad \Rightarrow \quad f^{(14)}(0) = -\dfrac{14!}{7!}$

**9**. Find the angle between the vectors $\vec{u} = \langle 1,1,0 \rangle$ and $\vec{v} = \langle 1,2,1 \rangle$.

**a**. $0°$

**b**. $30°$

**c**. $45°$

**d**. $60°$

**e**. $90°$

$|\vec{u}| = \sqrt{1+1} = \sqrt{2} \qquad |\vec{v}| = \sqrt{1+4+1} = \sqrt{6} \qquad \vec{u} \bullet \vec{v} = 1 + 2 = 3$

$\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{3}{\sqrt{2}\sqrt{6}} = \dfrac{\sqrt{3}}{2} \qquad \theta = 30°$

**10**. Evaluate the integral $\displaystyle\int_0^{1/2} \frac{1}{1+x^3}\,dx$ as an infinite series.

**a**. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{3n} = 1 - \frac{1}{2^3} + \frac{1}{2^6} - \frac{1}{2^9} + \cdots$

**b**. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3n+1}\left(\frac{1}{2}\right)^{3n+1} = \frac{1}{2} + \frac{1}{4 \bullet 2^4} + \frac{1}{7 \bullet 2^7} + \frac{1}{10 \bullet 2^{10}} + \cdots$

**c**. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{2}\right)^{3n+1} = \frac{1}{2} - \frac{1}{4 \bullet 2^4} + \frac{1}{7 \bullet 2^7} - \frac{1}{10 \bullet 2^{10}} + \cdots$ correctchoice

**d**. $\displaystyle\sum_{n=0}^{\infty}(-1)^n(3n-1)\left(\frac{1}{2}\right)^{3n-1} = -2 - \frac{2}{2^2} + \frac{5}{2^5} - \frac{8}{2^8} + \cdots$

**e**. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3n-1}\left(\frac{1}{2}\right)^{3n-1} = -2 - \frac{1}{2 \bullet 2^2} + \frac{1}{5 \bullet 2^5} - \frac{1}{8 \bullet 2^8} + \cdots$

$\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ..........Geometric Series

$\displaystyle\frac{1}{1+x^3} = \sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n}$ .......... Substitute $x \to -x^3$

$$\int_0^{1/2} \frac{1}{1+x^3}\,dx = \int_0^{1/2}\sum_{n=0}^{\infty}(-1)^n x^{3n}\,dx = \sum_{n=0}^{\infty}(-1)^n\left[\frac{x^{3n+1}}{3n+1}\right]_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{2}\right)^{3n+1}$$

Part II: Work Out (points indicated below)

Show all your work. Partial credit will be given. You may not use a calculator.

**11**. (10 points) Consider the planes

$$P_1: \quad 2x - y + z = 1$$
$$P_2: \quad x + y - 3z = 2$$

**a**. (2 pts) Fill in the blanks:

A normal to the plane $P_1$ is $\overrightarrow{N_1} =$ ________________ $\overrightarrow{N_1} = \langle 2,-1,1\rangle$

A normal to the plane $P_2$ is $\overrightarrow{N_2} =$ ________________ $\overrightarrow{N_2} = \langle 1,1,-3\rangle$

**b**. (3 pts) Find a vector parallel to the line of intersection of the two planes.

$$\vec{v} = \overrightarrow{N_1} \times \overrightarrow{N_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 1 & -3 \end{vmatrix} = \langle 2,7,3\rangle$$

**c**. (3 pts) Find a point on the line of intersection of the two planes.

Set $z = 0$: $\quad 2x - y = 1 \quad x + y = 2$ Add: $\quad 3x = 3 \quad \Rightarrow \quad x = 1, y = 1$

A point is $(1,1,0)$.

**d**. (2 pts) Find parametric equations for the line of intersection of the two planes.

$(x,y,z) = (1,1,0) + t\langle 2,7,3\rangle$

$x = 1 + 2t \qquad y = 1 + 7t \qquad z = 3t$

**12**. (15 points) Let $f(x) = \ln x$.

**a**. (10 pts) Find the 3rd degree Taylor polynomial $T_3$ for $f(x)$ about $x = 2$.

$$f(x) = \ln x \qquad f'(x) = \frac{1}{x} \qquad f''(x) = \frac{-1}{x^2} \qquad f'''(x) = \frac{2}{x^3}$$
$$f(2) = \ln 2 \qquad f'(2) = \frac{1}{2} \qquad f''(2) = \frac{-1}{4} \qquad f'''(2) = \frac{1}{4}$$

$$T_3 = f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \frac{1}{3!}f'''(2)(x-2)^3$$
$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3$$

**b**. (5 pts) If this polynomial $T_3$ is used to approximate $f(x)$ on the interval $1 \le x \le 3$, estimate the maximum error $|R_3|$ in this approximation using Taylor's Inequality.

$$|R_n(x)| < \frac{M}{(n+1)!}|x-2|^{n+1} \quad \text{where } M \ge \left|f^{(n+1)}(x)\right| \text{ for } 1 \le x \le 3.$$

Note: $n = 3$ and $n + 1 = 4$.

On the interval $1 \le x \le 3$, $\left|f^{(4)}(x)\right| = \left|\frac{-6}{x^3}\right|$ is largest at $x = 1$. So $M = \left|f^{(4)}(1)\right| = 6$.

On the interval $1 \le x \le 3$, $|x-2|$ is largest at $x = 1$ or $3$. So $|x-2| \le |3-2| = 1$. Therefore

$$|R_3| < \frac{M}{4!}|x-2|^4 \le \frac{6}{24} \bullet 1^4 = \frac{1}{4}$$

**13**. (15 points) Consider the points

$$P = (1,0,-1), \quad Q = (2,3,1) \quad \text{and} \quad R(0,4,1)$$

**a**. (5 pts) Find a vector orthogonal to the plane determined by $P$, $Q$ and $R$.

$$\vec{u} = \overrightarrow{PQ} = Q - P = \langle 1,3,2\rangle \qquad \vec{v} = \overrightarrow{PR} = R - P = \langle -1,4,2\rangle$$

$$\vec{N} = \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 2 \\ -1 & 4 & 2 \end{vmatrix} = \langle -2,-4,7\rangle$$

**b**. (5 pts) Find the area of the triangle with vertices $P$, $Q$ and $R$.

$$A = \frac{1}{2}|\vec{u}\times\vec{v}| = \frac{1}{2}\sqrt{4+16+49} = \frac{1}{2}\sqrt{69}$$

**c**. (5 pts) Find the equation of the plane determined by $P$, $Q$ and $R$.

$$N_1(x-x_0) + N_2(y-y_0) + N_3(z-z_0) = 0$$

$$-2(x-1) - 4(y) + 7(z+1) = 0 \quad \text{or} \quad 2x + 4y - 7z = 9$$

**14**. (10 points) Find the radius of convergence and the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n\sqrt{n+1}}(x-2)^n$.

To find the radius, apply the ratio test:

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x-2|^{n+1}}{3^{n+1}\sqrt{n+2}}\frac{3^n\sqrt{n+1}}{|x-2|^n} = \frac{1}{3}|x-2| < 1 \quad\Rightarrow\quad |x-2| < 3 \quad\Rightarrow\quad R = 3$$

Convergent on $-1 < x < 5$.

Check endpoints:

At $x = -1$: $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n\sqrt{n+1}}(-3)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ is convergent by the alternating series test.

At $x = 5$: $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n\sqrt{n+1}}(3)^n = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$ is divergent by the integral test or as a $p$-series with $p = \frac{1}{2} < 1$.

So the interval of convergence is $-1 \le x < 5$.