MATH 152 FINAL EXAM Spring 2000

Sections 513,514 Solutions P. Yasskin

Multiple Choice: (4 points each)

1. Compute $\int_0^{\pi/2} x\cos(3x)\,dx$

a. $-\frac{\pi}{3}-\frac{1}{3}$

b. $-\frac{\pi}{6}-\frac{1}{3}$

c. $-\frac{\pi}{3}+\frac{1}{3}$

d. $-\frac{\pi}{6}-\frac{1}{9}$ correctchoice

e. $-\frac{\pi}{3}+\frac{1}{9}$

Integrate by parts with $\begin{array}{ll} u = x & dv = \cos(3x)\,dx \\ du = dx & v = \frac{1}{3}\sin(3x) \end{array}$

$$\int_0^{\pi/2} x\cos(3x)\,dx = \left[\frac{x}{3}\sin(3x) - \frac{1}{3}\int\sin(3x)\,dx\right]_0^{\pi/2} = \left[\frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x)\right]_0^{\pi/2}$$
$$= \frac{\pi}{6}\sin\left(\frac{3\pi}{2}\right) - \frac{1}{9}\cos(0) = -\frac{\pi}{6} - \frac{1}{9}$$

2. Compute $\lim_{n\to\infty}\frac{2^n}{1+3^n}$

a. 0 correctchoice

b. $\frac{1}{2}$

c. $\frac{1}{1-\frac{2}{3}}$

d. $\frac{\frac{1}{2}}{1-\frac{2}{3}}$

e. ∞

$$\lim_{n\to\infty}\frac{2^n}{1+3^n} = \lim_{n\to\infty}\frac{\frac{2^n}{3^n}}{\frac{1}{3^n}+1} = \frac{0}{1} = 0$$

3. Compute $\int_0^{\pi/2}\sin^3\theta\,d\theta$

a. $-\frac{2}{3}$

b. $-\frac{1}{3}$

c. 0

d. $\frac{1}{3}$

e. $\frac{2}{3}$ correctchoice

Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$. So:

$$\int_0^{\pi/2}\sin^3\theta\,d\theta = \int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\,d\theta = -\int_1^0(1-u^2)\,du$$
$$= -\left[u - \frac{u^3}{3}\right]_1^0 = -0 + \left[1-\frac{1}{3}\right] = \frac{2}{3}$$

4. Which formula will give the arclength of the curve $y = \sin x$ between $x = 0$ and $x = \pi$?

a. $L = \int_0^{\pi} 2\pi x\sqrt{1+\cos^2 x}\,dx$
b. $L = \int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$ correctchoice
c. $L = \int_0^{\pi} 2\pi \sin x\sqrt{1+\cos^2 x}\,dx$
d. $L = \int_0^{\pi} 2\pi x\sqrt{1+\sin^2 x}\,dx$
e. $L = \int_0^{\pi} \sqrt{1+\sin^2 x}\,dx$

$$\frac{dy}{dx} = \cos x \qquad L = \int_0^{\pi} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$$

5. Which initial value problem describes the solution to the following problem:
A 100 gal tank is initially filled with sugar water whose concentration is $0.05\frac{\text{lb sugar}}{\text{gal water}}$. Sugar is added to the tank at the rate of $2\frac{\text{lb}}{\text{hr}}$ and pure water is added at the rate of $3\frac{\text{gal}}{\text{hr}}$. The mixture is kept well mixed and drained at the rate of $3\frac{\text{gal}}{\text{hr}}$. Find the find the amount of sugar in the tank after t hours.

a. $\frac{dS}{dt} = 2 - 0.03S, \quad S(0) = 5$ correctchoice
b. $\frac{dS}{dt} = 0.1 - 0.15S, \quad S(0) = 5$
c. $\frac{dS}{dt} = 3S - 0.02, \quad S(0) = 0.05$
d. $\frac{dS}{dt} = 0.02 - 3S, \quad S(0) = 5$
e. $\frac{dS}{dt} = 0.02 - 0.03S, \quad S(0) = 0.05$

$$\frac{dS}{dt}\frac{\text{lb}}{\text{hr}} = 2\frac{\text{lb}}{\text{hr}} - \frac{S\text{ lb}}{100\text{ gal}}3\frac{\text{gal}}{\text{hr}} = 2 - 0.03S \qquad S(0) = 0.05\frac{\text{lb}}{\text{gal}}100\text{ gal} = 5$$

6. Find the solution of the differential equation $\frac{dy}{dx} = 2x(1+y^2)$ satisfying the initial condition $y(2) = 0$.

a. $y = \tan(x^2) + 2$
b. $y = \tan^2(x-2)$
c. $y = \tan(x^2 - 4)$ correctchoice
d. $y = \tan(x^2 + \arctan 2)$
e. $y = \tan^2(x) - \tan^2 2$

$$\int \frac{dy}{1+y^2} = \int 2x\,dx \qquad \arctan y = x^2 + C \qquad \arctan 0 = 4 + C \qquad y = \tan(x^2-4)$$

7. Compute $\int_1^2 \frac{1}{(x-2)^{2/3}}\,dx$

a. $-\infty$
b. -3
c. -1
d. 3 correctchoice
e. ∞

$$\int_1^2 \frac{1}{(x-2)^{2/3}}\,dx = \left[3(x-2)^{1/3}\right]_1^2 = 3(2-2)^{1/3} - 3(1-2)^{1/3} = 3$$

8. Compute $\lim_{x\to 0} \frac{\sin(2x) - 2x}{3x^3}$

a. $-\frac{1}{9}$
b. -4
c. $-\frac{4}{9}$ correctchoice
d. $-\frac{8}{9}$
e. $-\frac{4}{3}$

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{3x^3} = \lim_{x\to 0} \frac{\left[2x - \frac{(2x)^3}{3!} + \cdots\right] - 2x}{3x^3} = \lim_{x\to 0}\left[-\frac{(2x)^3}{3!3x^3} + \cdots\right] = -\frac{8}{6 \bullet 3} = -\frac{4}{9}$$

9. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)^2}(x-3)^n$.

a. 0
b. $\frac{1}{2}$ correctchoice
c. 2
d. $\frac{1}{3}$
e. 3

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{(n+2)^2}\,\frac{(n+1)^2}{2^n(x-3)^n}\right| = 2|x-3|$$

Convergent if

$$L = 2|x-3| < 1 \text{ or } |x-3| < \frac{1}{2}. \text{ So } R = \frac{1}{2}.$$

10. Which term is incorrect in the following partial fraction expansion?

$$\frac{-10x^2+5x^3-8x+1}{(x-1)(x-3)^2(x^2+2)} = \underbrace{\frac{A}{x-1}}_{\text{a.}} + \underbrace{\frac{B}{x-3}}_{\text{b.}} + \underbrace{\frac{D}{(x-3)^2}}_{\text{c.}} + \underbrace{\frac{Ex+F}{x^2+2}}_{\text{d.}}$$

e. They are all correct. correctchoice

A linear or linear to a power in the denominator gets a constant in the numerator.
A quadratic or quadratic to a power in the denominator gets a linear in the numerator.
So all terms are correct.

11. A vector $\vec{u}$ has length 3. A vector $\vec{v}$ has length 4. The angle between them is $60°$. Find $\vec{u} \bullet \vec{v}$.

a. 6 correctchoice
b. $\frac{1}{24}$
c. $\frac{\sqrt{3}}{24}$
d. 24
e. $6\sqrt{3}$

$$\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos\theta = 3 \bullet 4 \bullet \cos 60° = \frac{12}{2} = 6$$

12. Find an equation for the plane containing the two lines

$$L_1: \quad x = 3+3t \quad y = 1+4t \quad z = 2+5t$$
$$L_2: \quad x = 3+t \quad y = 1 \quad z = 2-t$$

a. $-4x-8y-4z = 10$
b. $-4x+8y-4z = 10$
c. $x-2y+z = 3$ correctchoice
d. $x+2y+z = 7$
e. $x+2y+z = 10$

$$\vec{v}_1 = (3,4,5) \qquad \vec{v}_2 = (1,0,-1) \qquad \vec{N} = \vec{v}_1 \times \vec{v}_2 = (-4,8,-4) \qquad P = (3,1,2)$$
$$\vec{N} \bullet (X-P) = 0 \qquad -4(x-3)+8(y-1)-4(z-2) = 0$$
$$-4x+8y-4z = -12 \qquad x-2y+z = 3$$

Work Out (13 points each)

Show all your work. Partial credit will be given. You may not use a calculator.

13. Compute $\int \frac{\sqrt{x^2-1}}{x}\,dx$

$x = \sec\theta \qquad dx = \sec\theta\tan\theta\,d\theta \qquad \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

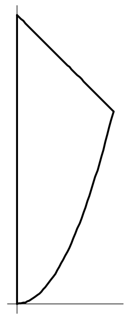
$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta \\ &= \int \tan^2\theta\,d\theta \\ &= \int 1 - \sec^2\theta\,d\theta \\ &= \theta - \tan\theta \\ &= \operatorname{arcsec} x - \sqrt{x^2-1} + C\end{aligned}$$

14. The parametric curve given by $x = t^2, \quad y = \frac{2}{3}t^3, \quad z = \frac{1}{4}t^4$ for $0 \le t \le 2$ is rotated about the y-axis. Find the area of the surface of revolution.
HINT: Factor the quantity in the square root.

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 2t^2, \qquad \frac{dz}{dt} = t^3$$

$$\begin{aligned}A &= \int_0^2 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_0^2 2\pi t^2\sqrt{(2t)^2 + (2t^2)^2 + (t^3)^2}\,dt \\ &= \int_0^2 2\pi t^2\sqrt{4t^2 + 4t^4 + t^6}\,dt \\ &= \int_0^2 2\pi t^2\sqrt{t^2(2+t^2)^2}\,dt = \int_0^2 2\pi t^3(2+t^2)\,dt = 2\pi\int_0^2 (2t^3 + t^5)\,dt \\ &= 2\pi\left[\frac{t^4}{2} + \frac{t^6}{6}\right]_0^2 \\ &= 2\pi\left[\frac{16}{2} + \frac{64}{6}\right] = 16\pi\left(1 + \frac{4}{3}\right) = \frac{112}{3}\pi\end{aligned}$$

15. The region in the first quadrant between the curves $y = x^2$ and $y = 6 - x$ is rotated about the y-axis. Find the volume of the solid of revolution.



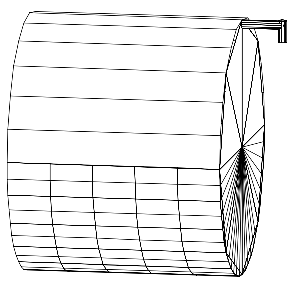
Use an x-integral with cylinders. $h = 6 - x - x^2$ $r = x$

To find the right endpoint, we solve $x^2 = 6 - x$, or $x^2 + x - 6 = 0$ or $(x-2)(x+3) = 0$. In the first quadrant $x = 2$.

$$V = \int_0^2 2\pi r h\,dx = \int_0^2 2\pi x(6 - x - x^2)\,dx = 2\pi \int_0^2 (6x - x^2 - x^3)\,dx$$

$$= 2\pi \left[3x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2 = 2\pi \left[12 - \frac{8}{3} - 4 \right]_0^2 = \frac{32}{3}\pi$$

16. A water tank has the shape of a circular cylinder laying on its side. It is 3 ft in radius and 5 ft long. It is half full of water. How much work is needed to pump the water out a spout at the top? (The weight density of water is $\rho g = 64.5 \frac{\text{lb}}{\text{ft}^3}$ but you may leave your answer as a multiple of ρg.)



We put the origin at the center of a circular end with y measured downward. So the water at height y must be lifted a distance $D = y + 3$.

To know the weight of a slab of water at height y, we must know its volume. Its length is 5. Its width is $2x$. Its thickness is dy. By the Pythagorean theorem $x = \sqrt{9 - y^2}$. So the weight is

$$dF = \rho g\,dV = \rho g 10 x\,dy = \rho g 10 \sqrt{9 - y^2}\,dy.$$

So the work is

$$W = \int D\,dF = \int_0^3 (y+3)\rho g 10 \sqrt{9 - y^2}\,dy$$

$$= 10\rho g \int_0^3 y\sqrt{9 - y^2}\,dy + 30\rho g \int_0^3 \sqrt{9 - y^2}\,dy$$

The first integral is a simple substitution. The second integral is the area of a quarter circle of radius 3.

$$W = 10\rho g \left[-\frac{1}{3}(9 - y^2)^{3/2} \right]_0^3 + 30\rho g \frac{1}{4}\pi (3)^2$$

$$= 10\rho g \frac{1}{3}(9)^{3/2} + \frac{135}{2}\rho g \pi = \rho g \left(90 + \frac{135}{2}\pi \right)$$

$$= 64.5 \left(90 + \frac{135}{2}\pi \right)$$