Student (Print) ______________________________ Section __________

Last, First Middle

Student (Sign) ______________________________

Student ID ______________________________

Instructor ______________________________

MATH 152
Exam2
Fall 2001
Test Form B
SOLUTIONS

1-10	/50
11	/10
12	/ 6
13	/ 6
14	/ 8
15	/10
16	/10
TOTAL	

Part I is multiple choice. There is no partial credit.
You may not use a calculator.

Part II is work out. Show all your work. Partial credit will be given.
You may use a calculator which cannot hold formulas.

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. Compute $\int_2^3 \frac{1}{(x-2)^{4/3}}\,dx$

- **a.** -3
- **b.** 3
- **c.** -1
- **d.** 1
- **e.** The integral diverges.   correctchoice

$$\int_2^3 \frac{1}{(x-2)^{4/3}}\,dx = \int_2^3 (x-2)^{-4/3}\,dx = -3(x-2)^{-1/3}\Big|_2^3 = -3 + 3\lim_{x\to 2^+}\frac{1}{(x-2)^{1/3}} = +\infty$$

2. For which values of p does the integral $\int_1^\infty x^{-p}\,dx$ converge?

- **a.** All values of p
- **b.** Only $p = 1$
- **c.** All values of $p > 1$ correctchoice
- **d.** All values of $p < 1$
- **e.** The integral diverges for all values of p

If $p \neq 1$, then $\int_1^\infty x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\Big|_1^\infty = \frac{1}{1-p}\frac{1}{x^{p-1}}\Big|_1^\infty = \begin{cases} \frac{1}{1-p} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$

If $p = 1$, then $\int_1^\infty x^{-p}\,dx = \int_1^\infty \frac{1}{x}\,dx = \ln|x|\Big|_1^\infty = \infty$

3. Compute the length of the arc $y = \frac{2}{3}x^{3/2}$ for $0 \le x \le 3$.

- **a.** $\frac{14}{3}$ correctchoice
- **b.** $\frac{\sqrt{7}}{3}$
- **c.** 7
- **d.** 14
- **e.** $\frac{16}{3}$

$$L = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+(x^{1/2})^2}\,dx = \int_0^3 \sqrt{1+x}\,dx = \frac{2(1+x)^{3/2}}{3}\Big|_0^3 = \frac{2(4)^{3/2}}{3} - \frac{2}{3} = \frac{14}{3}$$

4. Find the x-coordinate of the center of mass of a plate with constant density ρ bounded by the curves

$$x = 0, \qquad y = 0 \qquad \text{and} \qquad y = 8 - x^3$$

- **a.** $\frac{4}{5}$ correctchoice
- **b.** $\frac{5}{4}$
- **c.** $\frac{48}{5}\rho$
- **d.** $\frac{5}{48}$
- **e.** 12ρ

$$M = \rho \int_0^2 (8 - x^3)\, dx = \rho \left[8x - \frac{x^4}{4} \right]_0^2 = 12\rho$$
$$M_y = \rho \int_0^2 x(8 - x^3)\, dx = \rho \left[4x^2 - \frac{x^5}{5} \right]_0^2 = \rho \left(16 - \frac{32}{5} \right) = \frac{48}{5}\rho$$
$$\bar{x} = \frac{M_y}{M} = \frac{48\rho}{5 \cdot 12\rho} = \frac{4}{5}$$

5. If $y = f(x)$ is the solution to the differential equation $12\frac{dy}{dx} = xy^2$ which passes through the point $(0, 1)$, what is $f(4)$?

- **a.** $\frac{6}{5}$
- **b.** 3 correctchoice
- **c.** $\frac{12}{11}$
- **d.** 5
- **e.** $\frac{-1}{4}$

Separate variables: $\frac{12}{y^2}\, dy = x\, dx \quad \Rightarrow \quad \int \frac{12}{y^2}\, dy = \int x\, dx \quad \Rightarrow \quad \frac{-12}{y} = \frac{x^2}{2} + C$

Initial Condition: $x = 0, \; y = 1 \quad \Rightarrow \quad \frac{-12}{1} = \frac{0}{2} + C \quad \Rightarrow \quad C = -12$

So $\frac{-12}{y} = \frac{x^2}{2} - 12 \quad \Rightarrow \quad \frac{y}{-12} = \frac{1}{\frac{x^2}{2} - 12} = \frac{2}{x^2 - 24} \quad \Rightarrow \quad y = f(x) = \frac{-24}{x^2 - 24}$

$f(4) = \frac{-24}{16 - 24} = \frac{-24}{-8} = 3$

6. What is the integrating factor for the differential equation $\frac{dy}{dx} = 4xy + 5$?

- **a.** e^{2x^2}
- **b.** e^{-2x^2} correctchoice
- **c.** e^{-4x}
- **d.** $\frac{1}{x^4}$
- **e.** x^4

$P(x) = -4x \qquad I = e^{\int P(x)\, dx} = e^{\int -4x\, dx} = e^{-2x^2}$

7. Compute the limit of the sequence $a_n = \ln(2n+1) - \ln(n)$ for $n = 1, 2, \ldots$.

a. ∞
b. 1
c. $\ln 2$ correctchoice
d. 0
e. $-\ln 2$

$$\lim_{n\to\infty}[\ln(2n+1) - \ln(n)] = \lim_{n\to\infty} \ln\left(\frac{2n+1}{n}\right) = \ln\left(\lim_{n\to\infty} \frac{2n+1}{n}\right) = \ln 2$$

8. Let $I = \int_a^b f(x)\,dx$ and let T_n be the Trapezoid rule approximation of I with n intervals. If $f''(x) > 0$ for all x, which of the following is true?

a. $T_n > I$ correctchoice
b. $T_n < I$
c. $T_n = I$
d. $T_n > I$ for some values of n and $T_n < I$ for other values of n
e. $T_n = I$ for large values of n

Since $f''(x) > 0$, the graph is concave up (bending up). So the top edge of each trapezoid is above the curve. So $T_n > I$.

9. Which of the following integrals gives the area of the surface of revolution obtained by rotating the curve

$$x(t) = 3t - t^3, \qquad y(t) = 3t^2 \qquad \text{for} \quad 0 \le t \le 1$$

about the x-axis?

a. $2\pi \int_0^1 \frac{3t^2}{\sqrt{3 - 3t^2}}\,dt$
b. $6\pi \int_0^1 (3t - t^3)(t^2 + 1)\,dt$
c. $18\pi \int_0^1 (t^2 + t^4)\,dt$ correctchoice
d. $2\pi \int_0^1 \frac{3t - t^3}{\sqrt{3 - 3t}}\,dt$
e. $2\pi \int_0^1 \frac{3t^2}{\sqrt{(3t - t^3)^2 + (3t^2)^2}}\,dt$

$$A = 2\pi \int_0^1 y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi \int_0^1 3t^2\sqrt{(3 - 3t^2)^2 + (6t)^2}\,dt$$
$$= 2\pi \int_0^1 3t^2\sqrt{(9 - 18t^2 + 9t^4) + (36t^2)}\,dt = 2\pi \int_0^1 3t^2\sqrt{9 + 18t^2 + 9t^4}\,dt$$
$$= 2\pi \int_0^1 3t^2(3 + 3t^2)\,dt = 18\pi \int_0^1 (t^2 + t^4)\,dt$$

10. The sequence $a_n = \dfrac{(-1)^n}{n^2+2}$, for $n = 0, 1, 2, \ldots$ is

a. increasing and convergent.
b. decreasing and divergent.
c. increasing and divergent.
d. non-monotonic and convergent. correctchoice
e. non-monotonic and divergent.

The terms of the sequence are

$$\frac{1}{2}, \frac{-1}{3}, \frac{1}{6}, \frac{-1}{11}, \frac{1}{18}, \ldots$$

These are non-monotonic because they are alternating in sign.
However, they do converge to 0.

Part II: Work Out

Show all your work. Partial credit will be given.

You may use a calculator which cannot hold formulas.

11. (10 points) Solve the initial value problem

$$x\frac{dy}{dx} - 3y = x^2 \qquad \text{for } x > 0 \qquad \text{with} \quad y(1) = 0.$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad P(x) = -\frac{3}{x} \qquad \int P(x)\,dx = \int -\frac{3}{x}\,dx = -3\ln|x| = \ln x^{-3}$$

$$I = e^{\ln x^{-3}} = x^{-3} \qquad x^{-3}\frac{dy}{dx} - 3x^{-4}y = x^{-2} \qquad \frac{d}{dx}(x^{-3}y) = x^{-2}$$

$$x^{-3}y = \int x^{-2}\,dx = -x^{-1} + C \qquad y = -x^2 + Cx^3$$

$$x = 1 \quad y = 0\text{:} \qquad 1^{-3}0 = -1^{-1} + C \qquad C = 1$$

$$y = -x^2 + x^3$$

12. (6 points)

a. (3 pts) What is wrong with the following calculation?

$$\int_{-1}^{1} x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_{-1}^{1} = -\frac{2}{3}$$

b. (3 pts) What is the value of $\int_{-1}^{1} x^{-4}\,dx$?

a. The integrand becomes infinite at $x = 0$. So you need to break it into 2 pieces.
(2 pts for saying a negative value is impossible because x^{-4} is positive.)

b. $\int_{-1}^{1} x^{-4}\,dx = \int_{-1}^{0} x^{-4}\,dx + \int_{0}^{1} x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_{-1}^{0} + \left[\frac{x^{-3}}{-3}\right]_{0}^{1} = \infty + \infty = \infty$

13. (6 points) Use the Comparison Test to determine if $\int_1^\infty \frac{dx}{\sqrt{x^6+x}}$ converges or diverges.

You must clearly state (i) the integral you are using for comparison, (ii) whether it converges or diverges (You don't need to prove this.) and (iii) why this implies the original integral converges or diverges.

For large x, the term x^6 is larger than x. So compare to $\int_1^\infty \frac{dx}{\sqrt{x^6}} = \int_1^\infty \frac{dx}{x^3}$ which converges.

Now $x^6 + x > x^6$. So $\sqrt{x^6+x} > \sqrt{x^6} = x^3$. So $\frac{1}{\sqrt{x^6+x}} < \frac{1}{x^3}$. So $\int_1^\infty \frac{dx}{\sqrt{x^6+x}} < \int_1^\infty \frac{dx}{x^3}$. So $\int_1^\infty \frac{dx}{\sqrt{x^6+x}}$ also converges.

14. (8 points) What is the area of the surface generated when the arc of the circle $x^2 + y^2 = 100$ for $6 \le x \le 7$ is revolved about the x-axis?

$$y = \sqrt{100 - x^2} \qquad \frac{dy}{dx} = \frac{-x}{\sqrt{100-x^2}}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \frac{x^2}{100-x^2}}\,dx = \frac{10}{\sqrt{100-x^2}}\,dx$$

$$A = \int_6^7 2\pi y\,ds = 2\pi \int_6^7 \sqrt{100-x^2}\,\frac{10}{\sqrt{100-x^2}}\,dx = 2\pi \int_6^7 10\,dx = \Big[20\pi x\Big]_6^7 = 20\pi$$

15. (10 points)

a. (2 pts) Write a definite integral for the arclength of the parabola $y = \frac{x^2}{2}$ for $0 \le x \le 1$.

(Your answer should be $L = \int_0^1 \sqrt{1+x^2}\,dx$ but be sure to show your work.)

b. (4 pts) Use the trapezoid rule with $n = 5$ intervals to approximate this arclength.
(Use your calculator to evaluate your answer. Keep 5 decimal places.)

c. (1 pt) The arclength is $L = \int_0^1 g(x)\,dx$ where $g(x) = \sqrt{1+x^2}$. Show $g''(x) = \frac{1}{(1+x^2)^{3/2}}$.

d. (3 pts) Compute a bound on the error in the trapezoid rule approximation found in part (b) by using the trapezoid error bound formula

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad \text{where } K \ge |g''(x)| \text{ for } a \le x \le b$$

a. $\frac{dy}{dx} = x \qquad L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+x^2}\,dx$

b. $\Delta x = \frac{1}{5} = .2 \qquad f(x) = \sqrt{1+x^2}$

$$T_5 = \Delta x\left[\tfrac{1}{2}f(0) + f\left(\tfrac{1}{5}\right) + f\left(\tfrac{2}{5}\right) + f\left(\tfrac{3}{5}\right) + f\left(\tfrac{4}{5}\right) + \tfrac{1}{2}f(1)\right]$$

$$= .2\left(\tfrac{1}{2}(1) + \sqrt{1+.2^2} + \sqrt{1+.4^2} + \sqrt{1+.6^2} + \sqrt{1+.8^2} + \tfrac{1}{2}\sqrt{2}\right)$$

$$= .2(.5 + 1.01980 + 1.07703 + 1.16619 + 1.28062 + .5 \cdot 1.414213)$$

$$= 1.15015$$

c. $g(x) = \sqrt{1+x^2} \qquad g'(x) = \frac{1}{2}(1+x^2)^{-1/2}2x = x(1+x^2)^{-1/2}$

$$g''(x) = 1 \cdot (1+x^2)^{-1/2} + x \cdot \left[\frac{-1}{2}(1+x^2)^{-3/2}2x\right] = \frac{(1+x^2)-x^2}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}$$

d. $a = 0, \quad b = 1, \quad n = 5.$ The largest value of $g''(x)$ on $[0,1]$ is $K = g''(0) = 1.$ So

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{1(1-0)^3}{12(5)^2} = \frac{1}{300} = .00333$$

Note: $\int_0^1 \sqrt{1+x^2}\,dx = 1.1478$ and since $g''(x) > 0$, $T_5 > \int_0^1 \sqrt{1+x^2}\,dx$.

16. (10 points) A brine solution that contains 0.2 kg of salt per liter flows at a constant rate of $20\ \frac{\text{L}}{\text{min}}$ into a large tank that initially held 1000 L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at the constant rate of $20\ \frac{\text{L}}{\text{min}}$. Let $S(t)$ be the amount of salt in the tank at time t.

a. Set up the differential equation and initial condition for $S(t)$.

b. How much salt is in the tank after one hour?

a. $$\frac{dS}{dt}\frac{\text{kg}}{\text{min}} = \underbrace{0.2\frac{\text{kg}}{\text{L}} \cdot 20\frac{\text{L}}{\text{min}}}_{\text{in}} - \underbrace{\frac{S\ \text{kg}}{1000\ \text{L}} \cdot 20\frac{\text{L}}{\text{min}}}_{\text{out}}$$

$$\frac{dS}{dt} = 4 - \frac{1}{50}S \qquad S(0) = 0$$

b. Separable: $\dfrac{dS}{4 - \frac{1}{50}S} = dt \qquad \displaystyle\int \frac{dS}{4 - \frac{1}{50}S} = \int dt \qquad -50\ln\left|4 - \frac{1}{50}S\right| = t + C$

$\left|4 - \frac{1}{50}S\right| = e^{-t/50 - C/50} \qquad 4 - \frac{1}{50}S = Ae^{-t/50} \qquad \frac{1}{50}S = 4 - Ae^{-t/50} \qquad S = 200 - 50Ae^{-t/50}$

$t = 0 \quad S = 0: \qquad 4 - \frac{1}{50}0 = Ae^0 \qquad A = 4$

$S = 200 - 200e^{-t/50}$

$S(60) = 200 - 200e^{-6/5} \approx 139.76$

OR

Linear: $\dfrac{dS}{dt} + \dfrac{1}{50}S(t) = 4 \qquad I = e^{\int \frac{1}{50}dt} = e^{t/50}$

$e^{t/50}\dfrac{dS}{dt} + \dfrac{1}{50}e^{t/50}S(t) = 4e^{t/50} \qquad \dfrac{d}{dt}(e^{t/50}S) = 4e^{t/50}$

$e^{t/50}S = \displaystyle\int 4e^{t/50}\,dt = 200e^{t/50} + C \qquad S = 200 + Ce^{-t/50}$

$t = 0 \quad S = 0: \qquad 0 = 200 + Ce^0 \qquad C = -200$

$S = 200 - 200e^{-t/50}$

$S(60) = 200 - 200e^{-6/5} \approx 139.76$