Student (print) ______________________________ Section __________

Last First Middle

Student (sign) ______________________________

Student ID ______________________________

Instructor ______________________________

MATH 152
Exam 3
Fall 2001
Test Form B

Solutions

Part I is multiple choice. There is no partial credit.
You may not use a calculator.

Part II is work out. Partial credit will be given.
You may use a calculator that cannot hold formulas.

1–10	/50
11	/20
12	/10
13	/10
14	/10
total	

Part I: Multiple Choice (5 points each)
There is no partial credit. You may not use a calculator.

1. Find a unit vector perpendicular to $\vec{u} = \langle 4, 1, -1 \rangle$ and $\vec{v} = \langle 0, 1, 1 \rangle$.

A: $\langle 0, 0, 1 \rangle$ **B:** $\left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$ **C:** $\langle 1, -2, 2 \rangle$ **D:** $\left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$ **E:** $\left\langle \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right\rangle$

Correct choice: D

2. Find the sum of the series $\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n}$.

A: $\frac{4}{3}$ **B:** 6 **C:** $\frac{3}{4}$ **D:** The series diverges **E:** 12

Correct choice: E

3. Let $\{s_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$. If $s_n = \frac{n}{n+1}$ for $n = 1, 2, 3, \ldots$, find a_3.

A: $\frac{3}{4}$ **B:** $\frac{1}{6}$ **C:** $\frac{2}{3}$ **D:** $\frac{1}{12}$ **E:** 1

Correct choice: D

4. For which values of p does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge?

A: all $p \leq 1$ **B:** all $p < 1$ **C:** all $p \geq 1$ **D:** all $p > 1$ **E:** all $p > 0$

Correct choice: E

5. The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-2)^n (x-5)^n}{n+1}$ is

A: ∞ **B:** 2 **C:** 1 **D:** $\frac{1}{2}$ **E:** 0

Correct choice: D

6. Find $\lim_{x \to 0} \frac{\sin(x^2) - x^2}{x^6}$. [Hint: Use the Maclaurin series of $\sin(x^2)$.]

A: $\frac{1}{2}$ **B:** $-\frac{1}{2}$ **C:** $\frac{1}{6}$ **D:** $-\frac{1}{6}$ **E:** 0

Correct choice: D

7. Find the coefficient of $(x-8)^2$ in the Taylor series of $f(x) = x^{\frac{4}{3}}$ about $x = 8$.

A: $-\frac{1}{9}$ **B:** $-\frac{1}{6}$ **C:** $\frac{1}{18}$ **D:** 0 **E:** $\frac{2}{9}$

Correct choice: C

8. Assume the series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges for $x = 7$ and diverges for $x = 8$. Which of the following is true about the series?

A: The series converges at $x = -6$.
B: The series converges at $x = 0$.
C: The series diverges at $x = 2$.
D: The series diverges at $x = 0$.
E: We do not have enough information to decide since we don't know the c_n.

Correct choice: B

9. Let $f(x) = \frac{1}{2} e^{x^2}$. What is $f^{(2001)}(0)$?

A: $\dfrac{\frac{1}{2}}{(2001)!}$ **B:** $\frac{1}{2}(2001)!$ **C:** $\dfrac{(2001)!}{4000}$ **D:** 0 **E:** $\frac{1}{2}(-1)^{2001}(2001)!$

Correct choice: D

10. Which of the following series represents $\frac{1}{x}$ in some appropriate interval?

A: $\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

B: $\sum_{n=0}^{\infty}(x+1)^n$

C: $\sum_{n=0}^{\infty}\dfrac{(x-1)^n}{2^{n+1}}$

D: $\sum_{n=0}^{\infty}(x-1)^n$

E: $\sum_{n=0}^{\infty}(-1)^n(x+1)^n$

Correct choice: A

Part II: Work Out
Show all your work. Partial credit will be given.
You may use a calculator that cannot hold formulas.

11. (20 points = 5 points for each part)
For each of the following series determine whether the series converges or diverges. Justify your answers fully by giving the appropriate test and showing how it applies to the series.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{2n+1}$

Diverges because what is summed is not a null sequence: $\lim_{n\to\infty}\left|(-1)^n \frac{n+1}{2n+1}\right| = \frac{1}{2} \neq 0.$

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

Converges, by the comparison test:
we have $\frac{1}{2^n+n} < 2^{-n}$ and the geometric series $\sum_{n=1}^{\infty} 2^{-n}$ converges.

11. (continued)

(c) $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$

Converges,

(i) by the Leibniz criterion for alternating series:

we have $\left|\frac{(-2)^n}{n!}\right| \leq \frac{2^n}{2!\,3^{n-2}} = \frac{3^2}{2!}\left(\frac{2}{3}\right)^n$ for $n \geq 2$ and therefore $\lim_{n\to\infty}\left|\frac{(-2)^n}{n!}\right| = 0$;

or (ii) equals e^{-2}, as we know, and therefore it must converge;

or (iii) by the ratio test: $\left|\frac{(-2)^{n+1}}{(n+1)!}\Big/\frac{(-2)^n}{n!}\right| = \frac{2}{n+1} \to 0$ as $n \to \infty$.

(d) $\sum_{n=1}^{\infty} \frac{\ln(n)}{2n}$

Diverges because it dominates, for $n > 2$, the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{2n}$.

12. (10 points)
Consider the points $A(1,1,1)$, $B(0,2,0)$, $C(0,0,2)$.

(a) Show that the triangle ABC is isosceles. (6 points)

Sides have lengths

$$\overline{AB} = \sqrt{1+1+1} = \sqrt{3}\,, \quad \overline{BC} = \sqrt{0+4+4} = \sqrt{8}\,, \quad \overline{CA} = \sqrt{1+1+1} = \sqrt{3}\,,$$

so that A is equally distant from B and C.

(b) Compute the largest angle of the triangle ABC. (4 points)

Side BC is longest, so the largest angle is at corner A; use cosine theorem:

$$\overline{BC}^{\,2} = \overline{CA}^{\,2} + \overline{AB}^{\,2} - \overline{CA}\,\overline{AB}\,\cos\alpha\,,$$
$$8 = 3 + 3 - 2\cdot 3\cos\alpha\,,$$

so that

$$\cos\alpha = -\frac{1}{3}\,, \qquad \alpha = 109.47^\circ = 0.608\pi = 1.911\,.$$

13. (10 points)

Consider the series $\sum_{n=1}^{\infty} \frac{6}{(2n+1)(2n-1)}$.

(a) Compute the partial sums s_1, s_2, s_3. (3 points)

$$s_1 = \frac{6}{3 \cdot 1} = 2, \qquad s_2 = 2 + \frac{6}{5 \cdot 3} = \frac{12}{5}, \qquad s_3 = \frac{12}{5} + \frac{6}{7 \cdot 5} = \frac{18}{7}.$$

(b) Find a closed formula for s_n. [Hint: Use partial fractions to obtain a telescoping series.] (3 points)

$$s_n = \sum_{m=1}^{n} 3\left(\frac{1}{2m-1} - \frac{1}{2m+1}\right) = 3\left(1 - \frac{1}{2n+1}\right) = \frac{6n}{2n+1}.$$

(c) Use your answer to (b), or any other method, to decide whether the series converges or diverges. If it converges find its sum. (4 points)

Converges, since $s_n \to 3$ as $n \to \infty$; so the sum is 3.

14. (10 points = 5 points for each part)

(a) Find the Maclaurin series of $F(x) = \int_0^x e^{-t^2} dt$. (Write the general term.)

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$
$$= x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots$$

(b) Use this series to approximate $\int_0^{\frac{1}{4}} e^{-t^2} dt$ within 0.0005. Justify your answer.

The error is less than the modulus of the first term omitted (alternating series). For $x = \frac{1}{4}$, the first 3 terms are

$$x = \frac{1}{4} = 0.25000 > 0.0005\,,$$
$$\frac{1}{3}x^3 = \frac{1}{192} = 0.00521 > 0.0005\,,$$
$$\frac{1}{10}x^5 = \frac{1}{10240} = 0.00010 < 0.0005\,,$$

so

$$F\left(\frac{1}{4}\right) \simeq \frac{1}{4} - \frac{1}{192} = \frac{47}{192} = 0.2448$$

with an error not exceeding 0.00010. [Note: The actual error is 98.5% of the x^5 term.]