M152 Exam #3 Solutions
Spring 2002

1. Now $0 \leq \left| \frac{\sin 2n}{n} \right| \leq \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. By the squeeze theorem, $\lim_{n\to\infty} \frac{\sin 2n}{n} = 0$.

   The correct answer is **(d)**.

2. Now

   $$\langle\, 1/\sqrt{3}\,,\, -1/\sqrt{3}\,,\, -1/\sqrt{3}\,\rangle \cdot \mathbf{a} = (1/\sqrt{3})(1) + (-1/\sqrt{3})(-1) + (-1/\sqrt{3})(2) = 0$$
   $$\|\langle\, 1/\sqrt{3}\,,\, -1/\sqrt{3}\,,\, -1/\sqrt{3}\,\rangle\| = \sqrt{(1/\sqrt{3})^2 + (-1/\sqrt{3})^2 + (-1/\sqrt{3})^2} = \sqrt{(1/3) + (1/3) + (1/3)} = 1\ .$$

   The correct answer is **(b)**.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}}{5^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{3}{5}\right)^n = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n = \frac{1}{3}\left(\frac{(3/5)}{1-(3/5)}\right) = \frac{1}{2}\ .$

   The correct answer is **(e)**.

4. If $f(x) = \ln x$, then $f'(x) = 1/x$, $f''(x) = -1/(x^2)$ and $f'''(x) = 2/(x^3)$. Thus, $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$ and $f'''(1) = 2$. The third-degree Taylor polynomial for $f(x) = \ln x$ at $a = 1$ is

   $$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3\ .$$

   The correct answer is **(a)**.

5. $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n^2+2}{3n^2+1} = \frac{2}{3}\ .$

   The correct answer is **(b)**.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \lim_{j\to\infty}\sum_{n=1}^{j} \frac{1}{(n+2)(n+3)} = \lim_{j\to\infty}\sum_{n=1}^{j}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \lim_{j\to\infty}\left(\frac{1}{3} - \frac{1}{j+3}\right) = \frac{1}{3}\ .$

   The correct answer is **(a)**.

7. Now $0 \leq \frac{n}{n^3+5} < \frac{1}{n^2}$ for all $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+5}$ converges by the comparison test.

   The correct answer is **(c)**.

8. Now $0 = x^2 + y^2 + z^2 + 6x - 8y = (x+3)^2 - 9 + (y-4)^2 - 16 + z^2$, so $(x-(-3))^2 + (y-4)^2 + z^2 = 25$. The center of the sphere is located at $(-3\,,\,4\,,\,0)$.

   The correct answer is **(d)**.

9. If $f(x) = 1/(x(\ln x)^2)$, then $f(x) > 0$ and $f(x)$ is decreasing on $[2, \infty)$. Moreover,

$$\int_2^\infty f(x)\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{\ln x}\bigg|_2^b = \lim_{b\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln b}\right) = \frac{1}{\ln 2}\,.$$

The series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges by the integral test.

The correct answer is **(d)**.

10. $\displaystyle\sum_{n=1}^\infty \left|\frac{1}{n^{1.001}}\right| = \sum_{n=1}^\infty \frac{1}{n^{1.001}} = \sum_{n=1}^\infty \frac{1}{n^p}$, which converges since $p = 1.001 > 1$.

The correct answer is **(e)**.

11. Now

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x-3)^{n+1}}{(n+1)2^{n+1}}}{\dfrac{(-1)^n(x-3)^n}{n(2^n)}} \right| = \lim_{n\to\infty} \frac{n|x-3|}{2(n+1)} = \frac{|x-3|}{2} < 1\,,$$

when $|x-3| < 2$. Thus, the radius of convergence of the power series is $R = 2$.

When $x = 5$,

$$\sum_{n=1}^\infty \frac{(-1)^n(x-3)^n}{n(2^n)} = \sum_{n=1}^\infty \frac{(-1)^n}{n}\,,$$

which converges by the alternating series test, since $1/n > 0$ and $1/n \geq 1/(n+1)$ for all n and $\lim_{n\to\infty} 1/n = 0$.

When $x = 1$,

$$\sum_{n=1}^\infty \frac{(-1)^n(x-3)^n}{n(2^n)} = \sum_{n=1}^\infty \frac{1}{n}\,,$$

which is the harmonic series and diverges. The interval of convergence of the power series is $1 < x \leq 5$.

12. (a) The series $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}+1}$ has positive terms and diverges by the limit comparison test. Indeed,

$$\lim_{n\to\infty} \frac{\dfrac{1}{\sqrt{n}+1}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{1+(1/\sqrt{n})} = 1 > 0\,,$$

and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ diverges when $p = 1/2$.

(b) The series $\displaystyle\sum_{n=1}^\infty (-1)^n \frac{n}{n+1}$ diverges since $\displaystyle\lim_{n\to\infty} (-1)^n \frac{n}{n+1} \neq 0$. In fact, this limit doesn't exist.

13. (a) Now $1/(1-t) = \sum_{n=0}^{\infty} t^n$ for $|t| < 1$. Let $t = -x^2/4$. Then for $|x| < 2$,

$$\frac{1}{4+x^2} = \frac{1}{4}\left(\frac{1}{1+(x^2/4)}\right) = \frac{1}{4}\left(\frac{1}{1-t}\right) = \frac{1}{4}\sum_{n=0}^{\infty} t^n = \frac{1}{4}\sum_{n=0}^{\infty}(-x^2/4)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^{n+1}} .$$

The radius of convergence of this power series is $R = 2$.

(b) For $|x| < 2$,

$$\frac{d}{dx}\left(\frac{1}{4+x^2}\right) = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n x^{2n}}{4^{n+1}}\right)$$

$$-\frac{2x}{(4+x^2)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{4^{n+1}}$$

$$\frac{2x}{(4+x^2)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n)x^{2n-1}}{4^{n+1}} .$$

The radius of convergence of the differentiated power series is also $R = 2$.

14. (a) For all t,

$$\sin t = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

and

$$\sin(t^2) = \sum_{n=0}^{\infty}\frac{(-1)^n (t^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n t^{4n+2}}{(2n+1)!} .$$

Then

$$\int_0^x \sin(t^2)\,dt = \int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n t^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^x t^{4n+2}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)((2n+1)!)}x^{4n+3}$$

for all x. The Maclaurin series for $\int_0^x \sin(t^2)\,dt$ is $\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)((2n+1)!)}x^{4n+3}$.

(b) Using the first 2 terms of the series above,

$$\int_0^{0.1} \sin(t^2)\,dt \approx \sum_{n=0}^{1}\frac{(-1)^n}{(4n+3)((2n+1)!)}(0.1)^{4n+3} = \frac{(0.1)^3}{3} - \frac{(0.1)^7}{(7)(6)} .$$

The Maclaurin series for $\int_0^{0.1} \sin(t^2)\,dt$ is an alternating series, so

$$\left|\int_0^{0.1} \sin(t^2)\,dt - \sum_{n=0}^{1}\frac{(-1)^n}{(4n+3)((2n+1)!)}(0.1)^{4n+3}\right| < \frac{(0.1)^{11}}{(11)(5!)} .$$

15. The series $\sum_{n=1}^{\infty}[1-\cos(1/n)]$ has positive terms,

$$\lim_{n\to\infty}\frac{1-\cos(1/n)}{(1/n^2)} = \lim_{t\to 0}\frac{1-\cos t}{t^2} = \lim_{t\to 0}\frac{\sin t}{2t} = \lim_{t\to 0}\frac{\cos t}{2} = \frac{1}{2} > 0 ,$$

and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. Thus, $\sum_{n=1}^{\infty}[1-\cos(1/n)]$ converges by the limit comparison test.