Spring 2005 Math 152
Exam 1A: Solutions
Mon, 21/Feb ©2005, Art Belmonte

For specificity, lengths are in centimeters unless stated otherwise. Graphs appear at the bottom of the right column to conserve space.

1. (b) We have
$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)\,dx$
$= \left(-\frac{1}{2\sqrt{\pi}}\cos\left(x^2\right)\right)\Big|_0^{\sqrt{\pi}} = \left(\frac{1}{2}\cdot\frac{1}{\sqrt{\pi}}\right) - \left(-\frac{1}{2}\cdot\frac{1}{\sqrt{\pi}}\right) = \frac{1}{\sqrt{\pi}}$.

2. (d) Via Hooke's Law we have $F(x) = kx$ or $3 = k\left(\frac{1}{6}\right)$, whence $k = 18$. Thus $W = \int_0^2 18x\,dx = 9x^2\big|_0^2 = 36$ ft-lb.

3. (a) The area is $A = \int_0^2 \left|x^2-1\right| - 0\,dx$, computed thus:

$$\begin{aligned} A &= \int_0^1 1-x^2\,dx + \int_1^2 x^2-1\,dx \\ &= \left(x-\tfrac{1}{3}x^3\right)\Big|_0^1 + \left(\tfrac{1}{3}x^3 - x\right)\Big|_1^2 \\ &= \left(\tfrac{2}{3}\right) - (0) + \left(\tfrac{8}{3}-2\right) - \left(\tfrac{1}{3}-1\right) \\ &= \tfrac{4}{3} + \tfrac{8}{3} - 2 = 2\text{ cm}^2. \end{aligned}$$

4. (e) Use half-angle identities.

$$\begin{aligned} &\int_0^{\pi/2} \sin^2 x\cos^2 x\,dx \\ =\ &\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 2x)\cdot\tfrac{1}{2}(1+\cos 2x)\,dx \\ =\ &\tfrac{1}{4}\int_0^{\pi/2} 1-\cos^2 2x\,dx \\ =\ &\tfrac{1}{4}\int_0^{\pi/2} 1-\tfrac{1}{2}(1+\cos 4x)\,dx \\ =\ &\tfrac{1}{4}\cdot\tfrac{1}{2}\int_0^{\pi/2} 1-\cos 4x\,dx \\ =\ &\tfrac{1}{8}\left(x-\tfrac{1}{4}\sin 4x\right)\Big|_0^{\pi/2} = \tfrac{\pi}{16} - 0 = \tfrac{\pi}{16}. \end{aligned}$$

5. (c) We have
$dV = A\,dx = \pi r^2\,dx = \pi y^2\,dx = \pi\left(e^x\right)^2\,dx = \pi e^{2x}\,dx$.
Therefore
$V = \int_0^3 \pi e^{2x}\,dx = \frac{1}{2}\pi e^{2x}\big|_0^3 = \frac{\pi}{2}\left(e^6-1\right) \approx 621.13\text{ cm}^3$.

6. (a) First compute an antiderivative, then apply the FTC.

   - Let $\begin{array}{ll} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}\,dx & v = \frac{1}{2}x^2 \end{array}$. Then

     $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$.

   - Hence $\int_1^3 x\ln x\,dx = \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^3$
     $= \left(\frac{9}{2}\ln 3 - \frac{9}{4}\right) - \left(-\frac{1}{4}\right) = \frac{9}{2}\ln 3 - 2$.

7. (b) Use the trig identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned} \int \sin^3 x\cos^2 x\,dx &= \int \left(1-\cos^2 x\right)\cos^2 x\sin x\,dx \\ &= \int \left(\cos^4 x - \cos^2 x\right)(-\sin x)\,dx \\ &= \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C \end{aligned}$$

8. (e) We have
$\int_1^2 \frac{x}{x+1}\,dx = \int_1^2 1 - \frac{1}{x+1}\,dx = (x - \ln|x+1|)\big|_1^2$
$= (2-\ln 3) - (1-\ln 2) = 1 + \ln 2 - \ln 3 = 1 + \ln\frac{2}{3}$.

9. (d) Find where the curves intersect, draw a sketch, then set up and compute the integral for the volume.

   - Now $x^2 = 2x$ implies $x^2 - 2x = 0$ or $x(x-2) = 0$ whence $x = 0, 2$. (See figure at bottom of column.)
   - We have $dV = A\,dx = \left(\pi r_o^2 - \pi r_i^2\right)\,dx$
     $= \pi\left((2x)^2 - \left(x^2\right)^2\right)\,dx = \pi\left(4x^2 - x^4\right)\,dx$.
   - The volume is
     $V = \pi\int_0^2 4x^2 - x^4\,dx = \pi\left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2$
     $= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = 32\pi\left(\frac{5-3}{15}\right) = \frac{64}{15}\pi \approx 13.40\text{ cm}^3$.

10. (c) First compute an antiderivative, then apply the FTC.

   - Let $\begin{array}{ll} u = x & dv = f''(x)\,dx \\ du = dx & v = f'(x) \end{array}$. Then

     $\int xf''(x)\,dx = xf'(x) - \int f'(x)\,dx = xf'(x) - f(x)$.

   - Hence $\int_1^5 xf''(x)\,dx = \left(xf'(x) - f(x)\right)\big|_1^5$
     $= \left(5f'(5) - f(5)\right) - \left(f'(1) - f(1)\right)$
     $= \left(5(2) - 3\right) - (-1-4) = 7 + 5 = 12$.

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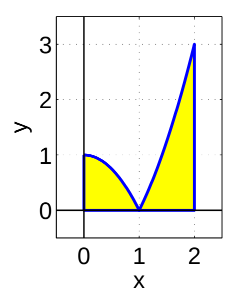

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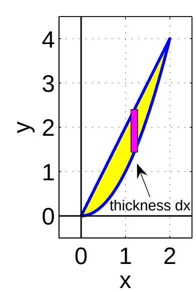

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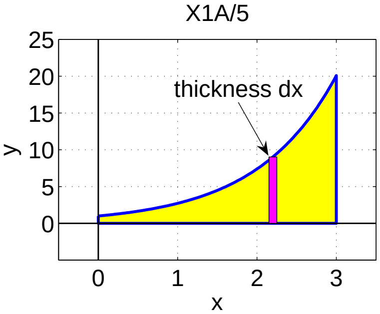


11. A diagram of the tank, its semicircular end, and a rectangular differential layer of water appears at bottom right.

- Clearly the layer is 3 m long, but how wide is it? Look at a diagram of the semicircular end of the tank. The width of the layer is $2y = 2\sqrt{1^2 - z^2} = 2\sqrt{1 - z^2}$.
- The rectangular layer of water has an area of

$$A = LW = 3\left(2\sqrt{1-z^2}\right) = 6\sqrt{1-z^2}.$$

Its thickness is dz. Here are the volume of the layer, its weight, and the work required to lift it to the top of the tank. Recall that $\delta = \rho g = 9800 \text{ N/m}^3$.

$$\begin{aligned} dV = A\,dz &= 6\sqrt{1-z^2}\,dz \\ dF = \delta\,dV &= 9800\,(6)\,\sqrt{1-z^2}\,dz \\ dW = dF\,\,(0 - z) &= -9800\,(6)\,z\sqrt{1-z^2}\,dz \end{aligned}$$

- The work required to pump the water out of the tank is

$$\begin{aligned} W &= \int dW = \int_{-1}^{0} -9800\,(6)\,z\left(1-z^2\right)^{1/2}\,dz \\ &= (9800)\,(6)\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(1-z^2\right)^{3/2}\Big|_{-1}^{0} \\ &= (9800)\,(2) - 0 = 19{,}600 \text{ J}. \end{aligned}$$

12. The volume of a differential cross-section is

$$dV = A\,dy = s^2\,dy = (2x)^2\,dy = 4x^2\,dy = 4(1 - \tfrac{1}{4}y^2)\,dy.$$

Hence the total volume is $\int_{-2}^{2} 4 - y^2\,dy = 2\int_0^2 4 - y^2\,dy$

$$= 2\left(4y - \tfrac{1}{3}y^3\right)\Big|_0^2 = 2\left(8 - \tfrac{8}{3}\right) - 0 = \tfrac{32}{3} \approx 10.67 \text{ cm}^3.$$

13. Find where the curves intersect, draw a sketch, then set up and compute the integral for the volume.

- Now $\sqrt{x} = 2 - x$ implies $x = 4 - 4x + x^2$ or $x^2 - 5x + 4 = (x-1)(x-4) = 0$, whence $x = 1, 4$. Toss out $x = 4$. See the figure in the next column.
- Via cylindrical shells, we have $V = \int 2\pi r h\,dy$.

$$\begin{aligned} V &= \int_0^1 2\pi y\big((2-y) - (y^2)\big)\,dy \\ &= 2\pi \int_0^1 2y - y^2 - y^3\,dy \\ &= 2\pi\left(y^2 - \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\right)\Big|_0^1. \\ &= 2\pi\left(1 - \tfrac{1}{3} - \tfrac{1}{4}\right) - 0 = \frac{5\pi}{6} \approx 2.62 \text{ cm}^3. \end{aligned}$$

14. Let $x = 5\sin\theta$. Then $dx = 5\cos\theta\,d\theta$. Hence (see figure at bottom of the next column)

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{25-x^2}} &= \int \frac{5\cos\theta\,d\theta}{25\sin^2\theta \cdot 5\cos\theta} \\ &= \tfrac{1}{25}\int \csc^2\theta\,d\theta \\ &= -\tfrac{1}{25}\cot\theta + C \\ &= -\frac{\sqrt{25-x^2}}{25x} + C \end{aligned}$$

15. We'll integrate the rational function via partial fractions.

- Split the integrand into a sum of partial fractions.

$$\begin{aligned} \frac{10}{(x-1)\left(x^2+9\right)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \\ 10 &= A\left(x^2+9\right) + (Bx+C)\,(x-1) \\ 0x^2 + 0x + 10 &= (A+B)\,x^2 + (C-B)\,x + (9A - C) \end{aligned}$$

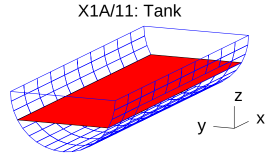
- Thus $A + B = 0$, $C - B = 0$, and $9A - C = 10$. Adding the first two equations we obtain $A + C = 0$, whence $C = -A$. Substituting this into the third equation gives $10A = 10$ whence $A = 1$, $C = -1$, and $B = C = -1$. Therefore,

$$\frac{10}{(x-1)\left(x^2+9\right)} = \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}.$$

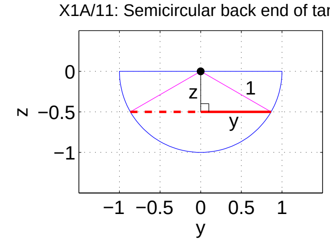
- Now integrate term-by-term.

$$\begin{aligned} &\int \frac{10}{(x-1)\left(x^2+9\right)}\,dx \\ &= \int \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\,dx \\ &= \ln|x-1| - \tfrac{1}{2}\ln\left(x^2+9\right) - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) + C \end{aligned}$$

X1A/11: Tank



X1A/11: Semicircular back end of tank



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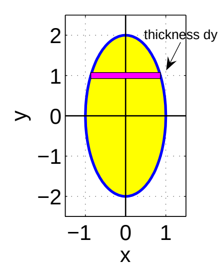

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