

Name _____ Sec _____ ID _____

MATH 152

Final Exam

Fall 2007

Sections 513 - 515

Solutions

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Multiple Choice: (6 points each)

1-11	/66
12	/12
13	/15
14	/10
Total	/103

1. A plate is bounded by the curves $y = x^2$, $y = -x^2$ and $x = 2$ measured in meters. Find the total mass of the plate if the surface density is $\rho = 3 \text{ kg/m}^2$.

- a. $\frac{8}{3}$
- b. $\frac{16}{3}$
- c. $\frac{32}{3}$
- d. 8
- e. 16 correct choice

$$M = \rho \int_0^2 (f - g) dx = 3 \int_0^2 (x^2 - -x^2) dx = 3 \left[\frac{2x^3}{3} \right]_0^2 = 16$$

2. A plate is bounded by the curves $y = x^2$, $y = -x^2$ and $x = 2$ measured in meters. Find the center of mass of the plate if the surface density is $\rho = 3 \text{ kg/m}^2$.

- a. $(\frac{3}{4}, 0)$
- b. $(\frac{3}{4}, \frac{1}{10})$
- c. $(\frac{3}{4}, \frac{6}{5})$
- d. $(\frac{3}{2}, 0)$ correct choice
- e. $(\frac{3}{2}, \frac{6}{5})$

$$M_y = \rho \int_0^2 x(f - g) dx = 3 \int_0^2 x(x^2 - -x^2) dx = 3 \left[\frac{2x^4}{4} \right]_0^2 = 24$$

$$\bar{x} = \frac{M_y}{M} = \frac{24}{16} = \frac{3}{2} \quad \bar{y} = 0 \text{ by symmetry.}$$

3. Compute $\int x^2 e^{2x} dx$.

- a. $\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C$
- b. $\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} + C$ correct choice
- c. $\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} - \frac{1}{2} e^{2x} + C$
- d. $\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{2} e^{2x} + C$
- e. $\frac{x^2}{2} e^{2x} - \frac{x}{4} e^{2x} - \frac{1}{4} e^{2x} + C$

$$\begin{aligned} u &= x^2 & dv &= e^{2x} dx \\ du &= 2x dx & v &= \frac{1}{2} e^{2x} \end{aligned} \quad \int x^2 e^{2x} dx = \frac{x^2}{2} e^{2x} - \int x e^{2x} dx$$

$$\begin{aligned} u &= x & dv &= e^{2x} dx \\ du &= dx & v &= \frac{1}{2} e^{2x} \end{aligned} \quad \begin{aligned} \int x^2 e^{2x} dx &= \frac{x^2}{2} e^{2x} - \left(\frac{x}{2} e^{2x} - \frac{1}{2} \int e^{2x} dx \right) \\ &= \frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} + C \end{aligned}$$

4. The parametric curve $x = 2t^2$ $y = t^3$ for $0 \leq t \leq 1$ is rotated about the y -axis. Which integral gives the area of the surface swept out?

- a. $2\pi \int_0^1 t \sqrt{16 + 9t^2} dt$
- b. $2\pi \int_0^1 t^3 \sqrt{16 + 9t^2} dt$
- c. $4\pi \int_0^1 t^3 \sqrt{16 + 9t^2} dt$ correct choice
- d. $2\pi \int_0^1 t^4 \sqrt{16 + 9t^2} dt$
- e. $4\pi \int_0^1 t^4 \sqrt{16 + 9t^2} dt$

$$\begin{aligned} A &= \int 2\pi r ds = \int_0^1 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^1 2\pi 2t^2 \sqrt{(4t)^2 + (3t^2)^2} dt \\ &= 4\pi \int_0^1 t^2 \sqrt{16t^2 + 9t^4} dt = 4\pi \int_0^1 t^3 \sqrt{16 + 9t^2} dt \end{aligned}$$

5. Which term appears in the partial fraction expansion of $\frac{3x^2 - 4x - 20}{(x - 2)^2(x^2 + 4)}$?

- a. $\frac{-2}{(x - 2)^2}$ correct choice
- b. $\frac{1}{(x - 2)^2}$
- c. $\frac{2}{(x - 2)^2}$
- d. $\frac{-2}{(x - 2)}$
- e. $\frac{1}{(x - 2)}$

$$\frac{3x^2 - 4x - 20}{(x - 2)^2(x^2 + 4)} = \frac{A}{(x - 2)} + \frac{B}{(x - 2)^2} + \frac{Cx + D}{(x^2 + 4)}$$

$$3x^2 - 4x - 20 = A(x - 2)(x^2 + 4) + B(x^2 + 4) + (Cx + D)(x - 2)^2$$

$$x = 2: \quad 12 - 8 - 20 = 0 + B(8) + 0 \quad -16 = 8B \quad B = -2$$

6. Find the solution of the differential equation $x \frac{dy}{dx} = 2y + x^3$ satisfying the initial condition $y(1) = 2$.

- a. $y = x^3 - \frac{7}{4}x^2$
- b. $y = x^3 + x^2$ correct choice
- c. $y = 2x^3 - x^2$
- d. $y = \frac{9}{5x^2} + \frac{1}{5}x^3$
- e. $y = \frac{5}{3x^2} + \frac{1}{3}x^3$

Put the equation in standard linear form: $\frac{dy}{dx} - \frac{2}{x}y = x^2$

The integrating factor is: $I = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = x^{-2} = \frac{1}{x^2}$

Multiply by the integrating factor: $\frac{1}{x^2} \frac{dy}{dx} - \frac{2}{x^3}y = 1$

Rewrite and integrate: $\frac{d}{dx} \left(\frac{1}{x^2} y \right) = 1 \quad \frac{1}{x^2} y = \int 1 dx = x + C$

Apply the initial conditions $x = 1, y = 2$: $2 = 1 + C \quad C = 1$

Substitute back and solve for y : $\frac{1}{x^2} y = x + 1 \quad y = x^3 + x^2$

7. The region in the first quadrant bounded by the curves $y = x^2$, $y = 0$ and $x = 2$ is rotated about the y -axis. Find the volume of the solid swept out.
- $\frac{32}{5}\pi$
 - $\frac{64}{5}\pi$
 - 8π correct choice
 - 4π
 - 2π

This is a x -integral. A vertical slice rotates into a cylinder of radius $r = x$ and height $h = x^2$. So the volume is

$$V = \int_0^2 2\pi rh \, dx = \int_0^2 2\pi x x^2 \, dx = 2\pi \frac{x^4}{4} \Big|_0^2 = 8\pi$$

8. If you approximate $f(x) = \ln(x)$ on the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$ by its 3rd degree Taylor polynomial centered at $x = 1$, namely $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, then the Taylor Remainder Theorem says the error $|R_3(x)|$ is less than

Taylor Remainder Theorem:

If $T_n(x)$ is the n^{th} degree Taylor polynomial for $f(x)$ centered at $x = a$ then there is a number c between a and x such that the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{(n+1)}$$

- 1
- $\frac{1}{2}$
- $\frac{1}{4}$ correct choice
- $\frac{1}{8}$
- 0

Here $a = 1$, and $n = 3$, and $|R_3(x)| = \frac{|f^{(4)}(c)|}{4!} |x-1|^4$

We need the largest values of $|f^{(4)}(c)|$ and $|x-1|^4$ for c and x in $\left[\frac{1}{2}, \frac{3}{2}\right]$.

$$f'(x) = \frac{1}{x} = x^{-1} \quad f''(x) = -x^{-2} \quad f'''(x) = 2x^{-3} \quad f^{(4)}(x) = -6x^{-4}$$

The max of $|f^{(4)}(c)| = \frac{6}{c^4}$ occurs at $c = \frac{1}{2}$ and is $M = 6 \cdot 2^4$.

The max of $|x-1|^4$ occurs at $\frac{1}{2}$ or $\frac{3}{2}$ and is $\frac{1}{2^4}$.

$$\text{So } |R_n(x)| \leq \frac{6 \cdot 2^4}{4!} \frac{1}{2^4} = \frac{6}{24} = \frac{1}{4}$$

9. Compute $\lim_{x \rightarrow 0} \frac{\sin(x^2) - x^2}{e^{(x^3)} - 1 - x^3}$.

- a. $-\frac{1}{6}$
- b. $\frac{1}{6}$
- c. $\frac{2}{3}$
- d. $-\frac{1}{3}$ correct choice
- e. $\frac{1}{3}$

$$\begin{aligned} \sin(t) &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots & \sin(x^2) &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots \\ e^{(t)} &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots & e^{(x^3)} &= 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \dots \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin(x^2) - x^2}{e^{(x^3)} - 1 - x^3} = \lim_{x \rightarrow 0} \frac{-\frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots}{\frac{x^6}{2!} + \frac{x^9}{3!} + \dots} = \lim_{x \rightarrow 0} \frac{-\frac{1}{3!} + \frac{x^4}{5!} - \dots}{\frac{1}{2!} + \frac{x^3}{3!} + \dots} = -\frac{2!}{3!} = -\frac{1}{3}$$

10. Find the volume of the parallelepiped with edge vectors

$$\vec{a} = (-2, 2, 1), \quad \vec{b} = (3, 2, 4) \quad \text{and} \quad \vec{c} = (1, -2, 3)$$

- a. -26
- b. 26
- c. $\sqrt{26}$
- d. -46
- e. 46 correct choice

$$V = \left| \begin{vmatrix} -2 & 2 & 1 \\ 3 & 2 & 4 \\ 1 & -2 & 3 \end{vmatrix} \right| = |-2(6 + 8) - 2(9 - 4) + 1(-6 - 2)| = |-28 - 10 - 8| = |-46| = 46$$

11. If \vec{u} points Down and \vec{v} points North West, in which direction does $\vec{u} \times \vec{v}$ point?

- a. North East correct choice
- b. South
- c. South West
- d. South East
- e. Up

Hold your right hand with the fingers pointing down and the palm facing left of forward. Then your thumb points right of forward.

Work Out: (Points indicated. Part credit possible.)

12. (12 points) Compute $\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx$

$$x = 2 \sin \theta \quad dx = 2 \cos \theta d\theta$$

$$x = 0 \quad @ \quad \sin \theta = 0 \quad \text{or} \quad \theta = 0 \quad \quad x = 1 \quad @ \quad \sin \theta = \frac{1}{2} \quad \text{or} \quad \theta = \frac{\pi}{6}$$

$$\begin{aligned} \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx &= \int_0^{\pi/6} \frac{4 \sin^2 \theta}{\sqrt{4-4 \sin^2 \theta}} 2 \cos \theta d\theta = \int_0^{\pi/6} 4 \sin^2 \theta d\theta = 4 \int_0^{\pi/6} \frac{1 - \cos 2\theta}{2} d\theta \\ &= 2 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/6} = 2 \left[\frac{\pi}{6} - \frac{1}{2} \sin \frac{\pi}{3} \right] = \frac{\pi}{3} - \frac{\sqrt{3}}{2} \end{aligned}$$

13. (15 points) A cone of radius 4 and height 6 and vertex down is filled with water up to height 2. Find the work done to pump the water out the top. Give your answer as a multiple of ρg .

The slice at height y is a circle. So its area is $A = \pi r^2$.

The radius is found using similar triangles: $\frac{r}{y} = \frac{4}{6}$ or $r = \frac{2}{3}y$

So the area is $A = \pi \left(\frac{2}{3}y \right)^2 = \frac{4}{9} \pi y^2$

and the volume of the slice of thickness dy is $dV = \frac{4}{9} \pi y^2 dy$.

This slice is lifted a distance $D = 6 - y$. So the work is

$$\begin{aligned} W &= \int \rho g D dV = \int_0^2 \rho g (6 - y) \frac{4}{9} \pi y^2 dy = \frac{4}{9} \rho g \pi \int_0^2 (6y^2 - y^3) dy \\ &= \frac{4}{9} \rho g \pi \left[2y^3 - \frac{y^4}{4} \right]_0^2 = \frac{4}{9} \rho g \pi (16 - 4) = \frac{16}{3} \rho g \pi \end{aligned}$$

14. (10 points) Start from the Maclaurin series: $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

a. (6 pt) Find the Maclaurin series for $\frac{1}{(1+x)^2}$.

HINT: Differentiate both sides of the given series.

$$\frac{d}{dx} \left(\frac{1}{1+x} \right) = \frac{-1}{(1+x)^2} \quad \frac{d}{dx} \left(\sum_{n=0}^{\infty} (-1)^n x^n \right) = \sum_{n=0}^{\infty} (-1)^n n x^{n-1}$$

$$\frac{1}{(1+x)^2} = - \sum_{n=0}^{\infty} (-1)^n n x^{n-1} \quad \text{You can stop here.}$$

$$= \sum_{n=0}^{\infty} (-1)^{n-1} n x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1} = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k$$

b. (4 pt) Compute $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}} = \sum_{n=1}^{\infty} (-1)^{n-1} n \left(\frac{1}{2} \right)^{n-1} = \frac{1}{\left(1 + \frac{1}{2} \right)^2} = \frac{1}{\left(\frac{3}{2} \right)^2} = \frac{4}{9}$$