Name______________________________ Section___________

MATH 152 FINAL EXAM Spring 2014

Sections 549-551 Solutions P. Yasskin

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Multiple Choice: (14 problems, 4 points each)

1. Find the area between the curves $y = 2x^2$ and $y = 4x$.

a. $\frac{4}{3}$

b. $\frac{8}{3}$ CORRECT

c. $\frac{16}{3}$

d. $\frac{32}{3}$

e. $\frac{64}{3}$

Solution: $2x^2 = 4x$ $x^2 = 2x$ $x = 0, 2$ The line is above the parabola.

$$A = \int_0^2 (4x - 2x^2)\,dx = \left[2x^2 - 2\frac{x^3}{3}\right]_0^2 = 8 - \frac{16}{3} = \frac{8}{3}$$

2. Find the average value of the function $f(x) = e^x$ on the interval $[-1, 1]$.

a. $e + \frac{1}{e}$

b. $\frac{1}{2}\left(e + \frac{1}{e}\right)$

c. $e - \frac{1}{e}$

d. $\frac{1}{2}\left(e - \frac{1}{e}\right)$ CORRECT

e. 1

Solution: $f_{ave} = \frac{1}{1-(-1)} \int_{-1}^{1} e^x\,dx = \frac{1}{2}\left[e^x\right]_{-1}^{1} = \frac{e - \frac{1}{e}}{2}$

3. Compute $\int_{-\pi/2}^{\pi/2} \sin^4\theta \cos\theta \, d\theta$.

- **a.** -6
- **b.** $-\frac{2}{5}$
- **c.** $\frac{2}{5}$ CORRECT
- **d.** $\frac{2}{3}$
- **e.** 6

Solution: $u = \sin\theta$, $du = \cos\theta \, d\theta$

$$\int_{-\pi/2}^{\pi/2} \sin^4\theta \cos\theta \, d\theta = \int u^4 \, du = \left[\frac{u^5}{5} \right] = \left[\frac{\sin^5\theta}{5} \right]_{-\pi/2}^{\pi/2} = \frac{2}{5}$$

4. Compute $\int x^2 e^{2x} \, dx$.

- **a.** $\left(\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)e^{2x} + C$
- **b.** $\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C$ CORRECT
- **c.** $\left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C$
- **d.** $\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{2}\right)e^{2x} + C$
- **e.** $\left(\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{2}\right)e^{2x} + C$

Solution: Use parts twice: $u = x^2$, $dv = e^{2x} \, dx$, $du = 2x \, dx$, $v = \frac{1}{2}e^{2x}$; $u = x$, $dv = e^{2x} \, dx$, $du = dx$, $v = \frac{1}{2}e^{2x}$

$$\int x^2 e^{2x} \, dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x} \, dx = \frac{1}{2}x^2 e^{2x} - \left(\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x} \, dx\right) = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C$$

5. Compute $\int \frac{\sqrt{x^2 - 1}}{x} \, dx$. HINT: $\tan^2\theta = \sec^2\theta - 1$

- **a.** $\operatorname{arcsec} x - \sqrt{x^2 - 1} + C$
- **b.** $\sqrt{x^2 - 1} + \operatorname{arcsec} x + C$
- **c.** $\sqrt{x^2 - 1} - \operatorname{arcsec} x + C$ CORRECT
- **d.** $\frac{1}{3}(x^2 - 1)^{3/2} - (x^2 - 1)^{1/2} + C$
- **e.** $\frac{1}{3}(x^2 - 1)^{3/2} + (x^2 - 1)^{1/2} + C$

Solution: $x = \sec\theta$, $dx = \sec\theta \tan\theta \, d\theta$

$$\int \frac{\sqrt{x^2 - 1}}{x} \, dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \sec\theta \tan\theta \, d\theta = \int \tan^2\theta \, d\theta$$

$$= \int \sec^2\theta - 1 \, d\theta = \tan\theta - \theta + C = \sqrt{x^2 - 1} - \operatorname{arcsec} x + C$$

6. In the partial fraction expansion, $\dfrac{4x^5+2x^2+8}{x^3(x^2+4)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+4} + \dfrac{Fx+G}{(x^2+4)^2}$, find C.

a. $\frac{1}{4}$

b. $\frac{1}{2}$ CORRECT

c. 2

d. 8

e. 16

Solution: Multiply by x^3: $\dfrac{4x^5+2x^2+8}{(x^2+4)^2} = Ax^2 + Bx + C + \dfrac{Dx+E}{x^2+4}x^3 + \dfrac{Fx+G}{(x^2+4)^2}x^3$

Set $x = 0$: $C = \dfrac{8}{16} = \dfrac{1}{2}$

7. Compute $\displaystyle\int_{-2}^{2} \frac{1}{x^3}\,dx$.

a. $-\frac{1}{4}$

b. $-\frac{1}{8}$

c. 0

d. $\frac{1}{4}$

e. undefined CORRECT

Solution: $\displaystyle\int_{-2}^{2} \frac{1}{x^3}\,dx = \int_{-2}^{0} \frac{1}{x^3}\,dx + \int_{0}^{2} \frac{1}{x^3}\,dx = \left[\frac{x^{-2}}{-2}\right]_{-2}^{0} + \left[\frac{x^{-2}}{-2}\right]_{0}^{2} = \left[\frac{-1}{2x^2}\right]_{-2}^{0} + \left[\frac{-1}{2x^2}\right]_{0}^{2}$

$\displaystyle= \left[\lim_{b\to 0^-}\left(\frac{-1}{2b^2}\right) - \left(\frac{-1}{8}\right)\right] + \left[\left(\frac{-1}{8}\right) - \lim_{a\to 0^+}\left(\frac{-1}{2a^2}\right)\right] = [-\infty] + [\infty]$ undefined

8. The region under the curve $y = \dfrac{1}{x^2+1}$ above the interval $[0,1]$ is revolved about the y-axis. Find the volume of the resulting solid.

a. $\pi\ln(2)$ CORRECT

b. $\pi\ln(2) - \pi$

c. $2\pi\ln(2)$

d. $4\pi\ln(2)$

e. $4\pi\ln(2) - 4\pi$

Solution: $A = \displaystyle\int_0^1 2\pi r h\,dx$ $\quad r = x \quad h = \dfrac{1}{x^2+1} \quad u = x^2+1 \quad du = 2x\,dx$

$\displaystyle= \int_0^1 2\pi x \frac{1}{x^2+1}\,dx = \int \pi \frac{1}{u}\,du = \pi\ln u = \left[\pi\ln(x^2+1)\right]_0^1 = \pi\ln(2)$

9. Find the arclength of the curve $x = 2t^4$, $y = t^6$ between $t = 0$ and $t = 1$.

a. $\frac{1}{27}$

b. $\frac{1}{3}$

c. $\frac{61}{12}$

d. $\frac{61}{27}$ CORRECT

e. $\frac{61}{54}$

Solution: $L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(8t^3)^2 + (6t^5)^2}\, dt$

$= \int_0^1 \sqrt{64t^6 + 36t^{10}}\, dt = \int_0^1 2t^3\sqrt{16 + 9t^4}\, dt \qquad u = 16 + 9t^4 \qquad du = 36t^3\, dt \qquad \frac{du}{36} = t^3\, dt$

$= \frac{1}{18}\int_{16}^{25} \sqrt{u}\, du = \left[\frac{1}{18}\frac{2u^{3/2}}{3}\right]_{16}^{25} = \frac{1}{27}(25^{3/2} - 16^{3/2}) = \frac{1}{27}(125 - 64) = \frac{61}{27}$

10. The curve $x = 2t^4$, $y = t^6$ between $t = 0$ and $t = 1$ is rotated about the x-axis. Which integral gives the area of the resulting surface?

a. $\int_0^1 4\pi t^9\sqrt{16 + 9t^4}\, dt$ CORRECT

b. $\int_0^1 8\pi t^7\sqrt{16 + 9t^4}\, dt$

c. $\int_0^1 4\pi t^7\sqrt{16 + 9t^4}\, dt$

d. $\int_0^1 2\pi t^7\sqrt{16 + 9t^4}\, dt$

e. $\int_0^1 4\pi t^3\sqrt{16 + 9t^4}\, dt$

Solution: $A = \int_{t=0}^1 2\pi r\, ds = \int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi t^6\sqrt{(8t^3)^2 + (6t^5)^2}\, dt$

$= \int_0^1 2\pi t^6\sqrt{64t^6 + 36t^{10}}\, dt = \int_0^1 4\pi t^9\sqrt{16 + 9t^4}\, dt$

11. The series $S = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{2^{2n}}$

a. converges to 4

b. converges to $\frac{4}{3}$

c. converges to $\frac{4}{7}$ CORRECT

d. converges to $\frac{2}{5}$

e. diverges

Solution: $S = \sum_{n=0}^{\infty}\left(\frac{-3}{4}\right)^n$ is geometric. $a = 1$ $\quad r = \left(\frac{-3}{4}\right)$ and $|r| < 1$

So $S = \frac{1}{1 - (-3/4)} = \frac{4}{4 + 3} = \frac{4}{7}$

12. Compute $\lim_{x\to 0} \frac{6x - x^3 - 6\sin x}{x^5}$

a. $\frac{6}{5!}$

b. -6

c. 6

d. $\frac{2}{3!}$

e. $-\frac{1}{20}$ CORRECT

Solution:

$$\lim_{x\to 0} \frac{6x - x^3 - 6\sin x}{x^5} = \lim_{x\to 0} \frac{6x - x^3 - 6\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)}{x^5} = \lim_{x\to 0} \frac{-6\left(\frac{x^5}{120} - \cdots\right)}{x^5} = -\frac{1}{20}$$

13. Compute $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n)!}$

a. $\frac{1}{2}$ CORRECT

b. -1

c. 0

d. 1

e. $\frac{1}{\sqrt{2}}$

Solution: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x$ So $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n} = \cos\frac{\pi}{3} = \frac{1}{2}$

14. Find the center and radius of the sphere $x^2 + 4x + y^2 + z^2 - 6z + 4 = 0$

a. center: $(-2, 0, 3)$ radius: $R = 2$

b. center: $(2, 0, -3)$ radius: $R = 9$

c. center: $(-2, 0, 3)$ radius: $R = 9$

d. center: $(2, 0, -3)$ radius: $R = 3$

e. center: $(-2, 0, 3)$ radius: $R = 3$ CORRECT

Solution: $(x^2 + 4x + 4) + y^2 + (z^2 - 6z + 9) - 9 = 0$ $(x+2)^2 + y^2 + (z-3)^2 = 9$
center: $(-2, 0, 3)$ radius: $R = 3$

Work Out (4 questions, Points indicated)
Show all you work.

15. (15 points) Consider the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$.

a. (3) Show the series is convergent.

Solution: The series alternates, $\frac{n}{2^{n-1}}$ decreases for $n > 2$ and $\lim_{n\to\infty} \frac{n}{2^{n-1}} = 0$.
So it converges by the alternating series test.

b. (6 Extra Credit) Show the series is absolutely convergent.
HINT: You may use these formulas without proof, if needed:
$2^{n-1} > n^3$ for $n \geq 12$ $\quad \int b^x\,dx = \frac{b^x}{\ln b} + C \quad \int x\,b^x\,dx = \frac{xb^x}{\ln b} - \frac{b^x}{\ln^2 b} + C$

Solution: The related absolute series is $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$.

Method 1: Apply the Ratio Test
$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2^n} \frac{2^{n-1}}{n} \right| = \frac{1}{2} < 1$ So the series is absolutely convergent.

Method 2: Apply the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Method 3: Apply the integral test with the integral $\int_1^{\infty} n \left(\frac{1}{2}\right)^{n-1} dx$.

c. (3) Compute S_7, the 7^{th} partial sum for S. Do not simplify.

Solution: $S_7 = \sum_{n=1}^{7} \frac{(-1)^{n-1} n}{2^{n-1}} = 1 - \frac{2}{2} + \frac{3}{4} - \frac{4}{8} + \frac{5}{16} - \frac{6}{32} + \frac{7}{64}$

d. (3) Find a bound on the remainder $|R_7| = |S - S_7|$ when S_7 is used to approximate S. Name the theorem you used.

Solution: $|R_7| < |a_8| = \frac{8}{2^7} = \frac{8}{128} = \frac{1}{16}$ by the alternating series bound theorem.

16. (20 points) Let $f(x) = \ln(x)$.

a. (6) Find the Taylor series for $f(x)$ centered at $x = 3$.

Solution:

$f(x) = \ln(x)$ | $f(3) = \ln(3)$

$f'(x) = \frac{1}{x}$ | $f'(3) = \frac{1}{3}$

$f''(x) = \frac{-1}{x^2}$ | $f''(3) = \frac{-1}{3^2}$

$f'''(x) = \frac{2}{x^3}$ | $f'''(3) = \frac{2}{3^3}$

$f^{(4)}(x) = \frac{-3!}{x^4}$ | $f^{(4)}(3) = \frac{-3!}{3^4}$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$ | $f^{(n)}(3) = \frac{(-1)^{n+1}(n-1)!}{3^n}$

$$T(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$$
$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{3^n n!}(x-3)^n$$
$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}(x-3)^n$$

b. (11) The Taylor series for $f(x)$ centered at $x = 4$ is $T(x) = 2\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}(x-4)^n$.

Find the interval of convergence for the Taylor series centered at $x = 4$. Give the radius and check the endpoints.

Solution: Apply the Ratio Test:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{4^{n+1}(n+1)} \frac{4^n n}{|x-4|^n} = \frac{|x-4|}{4} < 1 \qquad |x-4| < 4 \qquad \text{radius } R = 4.$$

endpoint: $x = 0$: $T(0) = 2\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}(-4)^n = 2\ln 2 - \sum_{n=1}^{\infty} \frac{1}{n}$

diverges (harmonic series)

endpoint: $x = 8$: $T(8) = 2\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}(4)^n = 2\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

converges (alternating series test) Interval of Convergence: $(0, 8]$

c. (3) If the cubic Taylor polynomial centered at $x = 4$ is used to approximate $\ln(x)$ on the interval $[3, 6]$, use the Taylor's Inequality to bound the error.

Taylor's Inequality:

Let $T_n(x)$ be the n^{th}-degree Taylor polynomial for $f(x)$ centered at $x = a$ and let $R_n(x) = f(x) - T_n(x)$ be the remainder. Then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

provided $M \ge |f^{(n+1)}(c)|$ for all c between a and x.

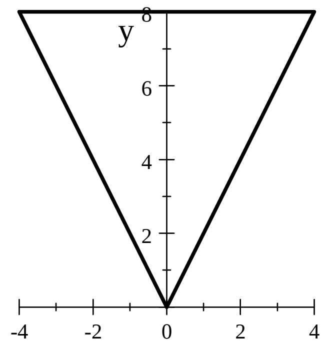
Solution: Here $n = 3$ and $a = 4$. Since c is between 4 and x, while x can be anything in $[3, 6]$, then c can be anything in $[3, 6]$.

$|f^{(4)}(c)| = \left|\frac{6}{c^4}\right|$ is largest on $[3, 6]$ when $x = 3$. So we take $M = \frac{6}{3^4} = \frac{2}{27}$

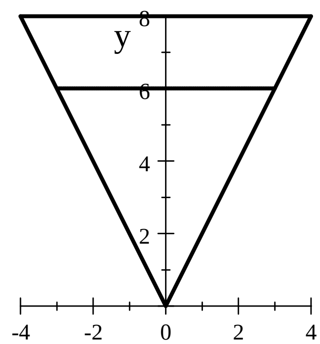
For x in $[3, 6]$, the largest value of $|x-4|$ is $|6-4| = 2$.

So $|R_3(x)| \le \frac{M}{4!}|x-4|^4 = \frac{2}{27 \cdot 4!}|x-4|^4 \le \frac{2}{27 \cdot 4!}2^4 = \frac{4}{81}$

17. (10 points) A conical tank with vertex down, radius 4, and height 8, is filled with water.
Find the work done to pump the water out the top of the tank.
Take the density of water to be ρ and the acceleration of gravity to be g.



Solution: Put the origin of the y-axis at the vertex of the cone at the bottom of the tank.



The slice at height y is a circle of radius r where $\frac{r}{y} = \frac{4}{8}$.
So $r = \frac{y}{2}$, the area is $A(y) = \pi r^2 = \frac{\pi}{4} y^2$,
the volume is $dV = A(y)\,dy = \frac{\pi}{4} y^2\,dy$
and the force is $dF = \rho g\,dV = \rho g \frac{\pi}{4} y^2\,dy$.
This slice needs to be lifted a distance $D = 8 - y$.
So the work is

$$W = \int_0^8 D\,dF = \int_0^8 (8-y)\,\rho g \frac{\pi}{4} y^2\,dy = \rho g \frac{\pi}{4} \int_0^8 (8y^2 - y^3)\,dy = \rho g \frac{\pi}{4} \left[\frac{8y^3}{3} - \frac{y^4}{4} \right]_0^8$$
$$= \rho g \frac{\pi}{4} \left(\frac{8^4}{3} - \frac{8^4}{4} \right) = \rho g \frac{\pi}{4} 8^4 \left(\frac{1}{3} - \frac{1}{4} \right) = \rho g \frac{\pi}{4} \frac{8^4}{12} = \rho g \pi \frac{256}{3}$$

18. (5 points) The region between the curves $y = 2x^2$ and $y = 4x$ is rotated about the x-axis.
Find the volume of the resulting solid.

Solution: x integral $\quad 2x^2 = 4x \quad x^2 = 2x \quad x = 0, 2$ The line is above the parabola.
Rotating about the x-axis gives thin washers.

$$V = \int_0^2 \pi (R^2 - r^2)\,dx = \int_0^2 \pi \left((4x)^2 - (2x^2)^2 \right) dx = \pi \int_0^2 (16x^2 - 4x^4)\,dx = \pi \left[16\frac{x^3}{3} - 4\frac{x^5}{5} \right]_0^2$$
$$= \pi \left[16\frac{2^3}{3} - 4\frac{2^5}{5} \right] = \pi 2^7 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2^8 \pi}{15}$$