MATH 152, Spring 2014
COMMON EXAM III - VERSION A

LAST NAME: Key FIRST NAME: ____________

INSTRUCTOR: ____________

SECTION NUMBER: ____________

UIN: ____________

DIRECTIONS:

1. The use of a calculator, laptop or computer is prohibited.
2. In Part 1 (Problems 1-15), mark the correct choice on your ScanTron using a No. 2 pencil. *For your own records, also record your choices on your exam!*
3. In Part 2 (Problems 16-20), present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
4. Be sure to *write your name, section number and version letter of the exam on the ScanTron form.*

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ____________

DO NOT WRITE BELOW!

| Question | Points Awarded | Points |
|---|---|---|
| 1-15 | | 60 |
| 16 | | 9 |
| 17 | | 9 |
| 18 | | 10 |
| 19 | | 6 |
| 20 | | 6 |
| | | 100 |

## PART I: Multiple Choice: 4 points each

1. If the integral test is used to determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges, we find that

(a) $\int_1^\infty xe^{-x^2}\,dx = \frac{2}{e}$, hence $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

(b) $\int_1^\infty xe^{-x^2}\,dx = \frac{e}{2}$, hence $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

(c) $\int_1^\infty xe^{-x^2}\,dx = 0$, hence $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

(d) $\int_1^\infty xe^{-x^2}\,dx = \frac{1}{2e}$, hence $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

(e) $\int_1^\infty xe^{-x^2}\,dx = \infty$, hence $\sum_{n=1}^{\infty} ne^{-n^2}$ diverges.

IT: $\int_1^\infty xe^{-x^2}\,dx$

$u = -x^2$

$du = -2x\,dx$

$= -\frac{1}{2}\int_1^\infty e^{-x^2}\,dx$

$= -\frac{1}{2}(e^{-\infty \to 0} - e^{-1}) = \frac{1}{2e} < \infty$

integral converges so does series.

2. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

(a) diverges because $0 \le \frac{\sqrt{n}}{n^2+1} \le \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ diverges by $p$ series.

(b) converges because $0 \le \frac{\sqrt{n}}{n^2+1} \le \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by $p$ series.

(c) converges because $0 \le \frac{\sqrt{n}}{n^2+1} \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by $p$ series.

(d) diverges because $0 \le \frac{1}{n} \le \frac{\sqrt{n}}{n^2+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by $p$ series.

(e) converges because $\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} = 0$.

$0 \le \frac{\sqrt{n}}{n^2+1} \le \frac{\sqrt{n}}{n^2} = \sum_{n=p}^{\infty} \frac{1}{n^{3/2}}$ which converges by p-series smaller series converges so does larger by comparison Test.

3. Find the center and radius of the sphere $x^2 + y^2 + z^2 = 2x - 3y + 4$.

(a) $C = \left(1, \frac{3}{2}, 0\right)$ and $r = \sqrt{\frac{29}{4}}$

(b) $C = \left(-1, \frac{3}{2}, 0\right)$ and $r = \sqrt{\frac{29}{4}}$

(c) $C = \left(1, -\frac{3}{2}, 0\right)$ and $r = \frac{29}{4}$

(d) $C = \left(1, \frac{3}{2}, 0\right)$ and $r = \frac{29}{4}$

(e) $C = \left(1, -\frac{3}{2}, 0\right)$ and $r = \sqrt{\frac{29}{4}}$

$x^2 - 2x + 1 + y^2 + 3y + \frac{9}{4} + z^2 = 4 + 1 + \frac{9}{4}$

$(x-1)^2 + (y + \frac{3}{4})^2 + z^2 = \frac{29}{4}$

$C = (1, -\frac{3}{2}, 0)$

$r = \sqrt{\frac{29}{4}}$

4. What is the interval and radius of convergence of the series $\sum_{n=1}^{\infty} n!(4x-1)^n$?

(a) $I = \left\{-\frac{1}{4}\right\}$, $R = 0$

(b) $I = \{-1\}$, $R = 0$

(c) $I = \left\{\frac{1}{4}\right\}$, $R = 0$

(d) $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$, $R = \frac{1}{8}$

(e) $I = (-\infty, \infty)$, $R = \infty$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,(4x-1)^{n+1}}{n!\,(4x-1)^n}\right|$$

$$= \lim_{n\to\infty} \left|(n+1)(4x-1)\right| = \infty$$

unless $x = \frac{1}{4}$.

Thus $R = 0$ and $I = \left\{\frac{1}{4}\right\}$

5. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

(a) converges absolutely by the $p$ series test.

(b) converges but not absolutely.

(c) converges by the Ratio Test.

(d) diverges by the $p$ series test.

(e) diverges by the Test for Divergence.

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

which converges by $p$-series. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ converges absolutely

6. Write $f(x) = \dfrac{x}{1+3x}$ as a power series. Include the radius of convergence.

(a) $f(x) = \sum_{n=0}^{\infty} 3^n x^{n+1}$, $R = \frac{1}{3}$

(b) $f(x) = \sum_{n=0}^{\infty} (-3)^n x^{n+1}$, $R = \frac{1}{3}$

(c) $f(x) = \sum_{n=0}^{\infty} 3^n x^{n+1}$, $R = 3$

(d) $f(x) = \sum_{n=0}^{\infty} (-3)^n x^{n-1}$, $R = \frac{1}{3}$

(e) $f(x) = \sum_{n=0}^{\infty} (-3)^n x^{n+1}$, $R = 3$

$x \sum_{n=0}^{\infty} (-3x)^n$, $\quad |-3x| < 1$

$|x| < \frac{1}{3}$.

$$\boxed{\sum_{n=0}^{\infty} (-3)^n x^{n+1} \qquad R = \frac{1}{3}}$$

7. Find the third degree Taylor polynomial for $f(x) = \sqrt{x}$ about $a = 4$.

(a) $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

(b) $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{1}{256}(x-4)^3$

(c) $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{3}{76}(x-4)^3$

(d) $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{3}{256}(x-4)^3$

(e) $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 - \frac{3}{512}(x-4)^3$

$f(x) = x^{\frac{1}{2}} \quad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \quad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \quad f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} \quad f'''(4) = \frac{3}{8(32)}$

$$T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{\cancel{3}}{8(32)} \frac{(x-4)^3}{\cancel{6}\,2}$$

$$T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

8. Using the known Maclaurin series for $\sin(x)$, find the Maclaurin series for $\sin(x^3)$.

(a) $\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$

(b) $\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

(c) $\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+1)!}$

(d) $\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$

(e) $\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n)!}$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

9. Describe the surface in three dimensional space: $x^2 + y^2 = 9$.

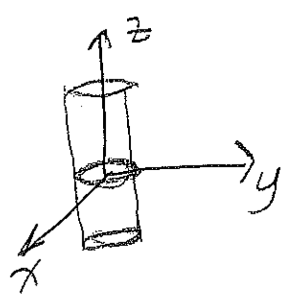
(a) Parabola

(b) Cylinder centered around the $z$ axis with radius 3

(c) Circle

(d) Hyperbola

(e) Sphere



10. If we use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n+1)!}$ converges or diverges, we find

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, thus $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n+1)!}$ diverges.

(b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$, thus $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n+1)!}$ converges absolutely.

(c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, thus $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n+1)!}$ converges absolutely.

(d) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty > 1$, thus $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n+1)!}$ diverges.

(e) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, thus the ratio test fails.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n}(-1)(n+1)\cancel{n!}\,\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}\,\cancel{(-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(n+1)}{(2n+3)(2n+2)} \right| = 0 < 1$$

converges abs.

11. Suppose we use $s_4$, the fourth partial sum, to approximate the sum of the series $S = \sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$. Which of the following is true from the Alternating Series Estimation Theorem regarding an upper bound of the absolute value of the remainder? ?

(a) $|S - S_4| = |R_4| < \frac{32}{120}$

(b) $|S - S_4| = |R_4| < -\frac{32}{120}$

(c) $|S - S_4| = |R_4| < \frac{16}{24}$

(d) $|S - S_4| = |R_4| < \frac{8}{6}$

(e) $|S - S_4| = |R_4| < \frac{32}{5}$

$$|R_4| \le |a_5| = \frac{32}{120}$$

12. Using Taylor's Inequality, which of the following gives a bound on the remainder if we used $T_3(x)$ centered at 2 to approximate $f(x) = \ln x$ on the interval $1 \le x \le 2.5$? Taylor's Inequality: $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$, where $|f^{(n+1)}(x)| \le M$ for $x$ in the interval $1 \le x \le 2.5$.

(a) $|f(x) - T_3(x)| = |R_3(x)| \le \frac{1}{64}$

(b) $|f(x) - T_3(x)| = |R_3(x)| \le \frac{96}{125}$

(c) $|f(x) - T_3(x)| = |R_3(x)| \le \frac{1}{4}$

(d) $|f(x) - T_3(x)| = |R_3(x)| \le \frac{1}{2}$

(e) $|f(x) - T_3(x)| = |R_3(x)| \le \frac{1}{8}$

$$R_3(x) = \frac{M}{4!}|x-2|^4$$

$$M = \max |f^4(x)| \quad 1 \le x \le 2.5$$

$$M = \max \left| \frac{-6}{x^4} \right| \quad 1 \le x \le 2.5$$

$\boxed{M = 6}$

$$|R_3(x)| \le \frac{6}{24}|x-2|^5$$

max distance from 2 is 1.

$f = \ln x$

$f' = \frac{1}{x}$

$f'' = -\frac{1}{\dots}$

$f''' = \frac{2}{x^3}$

$f^4 = -\frac{6}{x^4}$

$|R_3(x)| \le \frac{1}{\dots}$

13. Find all value(s) of $p$ for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges.

(a) $p > 1$
(b) $p > 0$
(c) $p \geq 1$
(d) $p \geq 0$
(e) The series converges for all real numbers $p$.

As long as $p > 0$, $\{\frac{1}{n^p}\}$ converges by AST

14. Find the sum of the series $\sum_{n=0}^{\infty} \frac{4(-3)^n}{n!}$

(a) $4e^3$
(b) 0
(c) $e^{-3}$
(d) $4e^{-3}$
(e) $\frac{1}{4}e^3$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$4e^{-3} = 4\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$$

15. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$

(a) $\frac{1}{2}$
(b) $\frac{\sqrt{3}}{2}$
(c) $-\frac{\sqrt{3}}{2}$
(d) $-\frac{1}{2}$
(e) 0

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos\left(\frac{\pi}{3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$$

$$= \frac{1}{2}$$

## PART II WORK OUT

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *Box your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (9 pts) Find the interval and radius of convergence of the series $\sum_{n=2}^{\infty} \frac{3^n(x-1)^n}{\ln n}$. Be sure to test the endpoints for convergence.

$$RT: \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-1)^{n+1}}{\ln(n+1)}\cdot\frac{\ln n}{3^n(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{3^n}\cdot 3\cancel{(x-1)^n}(x-1)}{\ln(n+1)}\cdot\frac{\ln n}{\cancel{3^n(x-1)^n}}\right|$$

note:

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{L}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{n+1}{n} = 1$$

$$= \lim_{n\to\infty}\left|\frac{3(x-1)\ln n}{n+1}\right|$$

$$= 3|x-1| < 1$$

$$|x-1| < \frac{1}{3} \qquad \boxed{R = \frac{1}{3}}$$

$\frac{2}{3}$ 1 $\frac{4}{3}$

Test endpoints:

$$① x = \frac{4}{3}: \sum_{n=2}^{\infty}\frac{3^n\left(\frac{1}{3}\right)^n}{\ln n} = \sum_{n=2}^{\infty}\frac{1}{\ln n} > \sum_{n=2}^{\infty}\frac{1}{n}$$

diverges by p-series

larger also diverges

$$② x = \frac{2}{3}: \sum_{n=2}^{\infty}\frac{3^n\left(-\frac{1}{3}\right)^n}{\ln n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$$

converges by Ast because $\frac{1}{\ldots} \le \frac{1}{n}$ & $\lim_{n\to\infty}\frac{1}{\ln n} = 0$

$\boxed{I = [\frac{2}{3}, \frac{4}{3})}$

17. (9 pts) Find a power series representation for $f(x) = \frac{1}{(2+x)^2}$. What is the associated radius of convergence?

$$\int \frac{1}{(2+x)^2}\,dx = -\frac{1}{2+x}$$

$$= -\frac{1}{2} \cdot \frac{1}{1+\frac{x}{2}}$$

$$= -\frac{1}{2} \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n, \quad \text{where } \left|-\frac{x}{2}\right| < 1$$

$$|x| < 2$$

$$\boxed{R = 2}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n x^n$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} x^n$$

Thus $$\frac{1}{(2+x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} x^n$$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n+1} n\, x^{n-1}$$

or $$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+2}(n+1)\, x^n$$

18. Consider the series $S = \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^4}$.

(i) (6 pts) Prove the series converges.

Integral test: $\int_2^\infty \frac{dx}{x(\ln x)^4}$

$u = \ln x$

$du = dx$

$\int \frac{du}{u^4} = \frac{-1}{3u^3}$

$= -\frac{1}{3(\ln x)^3}$

$\frac{1}{n(\ln n)^4} > 0$ and decreases

$= -\frac{1}{3(\ln x)^3} \Big|_2^\infty$

$= \cancel{-\frac{1}{3(\ln \infty)^3}} \to 0 + \frac{1}{3(\ln 2)^3}$

$= \frac{1}{3(\ln 2)^3} < \infty$

since improper integral converges, so does the series

(ii) (4 pts) Find $s_6$, that is the sum of the first 5 terms, to approximate the sum of the series and find a bound on the remainder.

$S_6 = \sum_{n=2}^{6} \frac{1}{n(\ln n)^4} = \frac{1}{2(\ln 2)^4} + \frac{1}{3(\ln 3)^4} + \ldots + \frac{1}{6(\ln 6)^4}$

$R_6 \leq \int_6^\infty \frac{dx}{x(\ln x)^4} = -\frac{1}{3(\ln x)^3}\Big|_6^\infty$

$= \frac{1}{3(\ln 6)^3}$

$\boxed{R_6 \leq \frac{1}{3(\ln 6)^3}}$

19. (6 pts) Find the Tayor series for $f(x) = e^{3x}$ about $a = 2$ (that is, centered at 2).

$$e^{3x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$f(x) = e^{3x}$$
$$f'(x) = 3e^{3x}$$
$$f''(x) = 3^2 e^{3x}$$
$$\vdots$$
$$f^{n}(x) = 3^n e^{3x}$$
$$f^{(n)}(2) = 3^n e^6$$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n e^6}{n!}(x-2)^n$$

20. (6 pts) Determine whether the series $\sum_{n=1}^{\infty} \frac{n^5 - n^4}{n^6+1}$ converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^5 - n^4}{n^6+1} \leq \sum_{n=1}^{\infty} \frac{1}{n},$$ which diverges by p-series, so CT Fails.

Limit comparison Test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^5 - n^4}{n^6+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^6 - n^5}{n^6+1} = 1 > 0$$

so Both series diverge since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.