

Name _____ Section _____

MATH 152

FINAL EXAM Version B

Spring 2016

Sections 555-557

Solutions

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1-13	/52
14	/20
15	/20
16	/5
17	/5
18	/5
Total	/107

Multiple Choice: (13 problems, 4 points each)

1.

Average Value of a Function

New Problem or Modify or Make Your Own Problem

Find the average value of the function $f(x) = \sin(x)$ on the interval $[a,b] = [0,\pi]$.

- a. $\frac{1}{\pi}$
- b. $\frac{2}{\pi}$ correct choice
- c. 1
- d. 2
- e. 2π

Solution: $f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{\pi} \int_0^{\pi} \sin(x) dx = -\frac{1}{\pi} \cos(x) \Big|_0^{\pi} = -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$

2.

Integrals Which are Improper at an Endpoint

New Problem

Problem Statement:

Determine if the following improper integral is convergent or divergent.

$$\int_{-2}^{\infty} (x+4)^{-\frac{1}{3}} dx$$

If convergent, compute it.
If divergent, determine if it is +infinity, -infinity, or neither.

- a. converges to $\frac{3}{2^{1/3}}$
- b. converges to $-\frac{3}{2^{1/3}}$
- c. diverges to $-\infty$
- d. diverges to ∞ correct choice
- e. diverges but not to $\pm\infty$

Solution: $\int_{-2}^{\infty} (x+4)^{-1/3} dx = \frac{3(x+4)^{2/3}}{2} \Big|_{-2}^{\infty} = \infty - \frac{3(2)^{2/3}}{2} = \infty$

Integration By Parts

3.

Use integration by parts
 to compute the integral:

$$J = \int_0^2 x e^{(-x)} dx$$

- a. $-e^{-2}$
- b. $-e^{-2} - 1$
- c. $1 - e^{-2}$
- d. $1 - 3e^{-2}$ correct choice
- e. $1 + e^{-2}$

Solution:

$u = x$	$dv = e^{-x} dx$
$du = dx$	$v = -e^{-x}$

$$J = \left[-xe^{-x} + \int e^{-x} dx \right]_0^2 = \left[-xe^{-x} - e^{-x} \right]_0^2$$

$$= (-2e^{-2} - e^{-2}) - (-0 - e^{-0}) = -3e^{-2} + 1$$

4.

Trigonometric Integrals

Use a substitution
 to compute the integral:

$$J = \int_0^{\frac{1}{2}\pi} \sin^3 x dx$$

- a. $-\frac{4}{3}$
- b. $-\frac{1}{4}$
- c. $\frac{1}{4}$
- d. $\frac{2}{3}$ correct choice
- e. $\frac{4}{3}$

Solution:

$u = \cos x$
$du = -\sin x dx$

$$J = \int_0^{\pi/2} (1 - \cos^2 x) \sin x dx = -\int_1^0 (1 - u^2) du = \left[-u + \frac{u^3}{3} \right]_1^0$$

$$= -\left(-1 + \frac{1}{3}\right) = \frac{2}{3}$$

5.

Integration by Trigonometric Substitution

New Integral

Goal: Evaluate the indefinite integral using a trigonometric substitution:

$$I = \int (x^2 + 16)^{-\frac{3}{2}} dx$$

- a. $\frac{1}{16} \int \csc^2 \theta d\theta$
- b. $\frac{1}{64} \int \sec^2 \theta d\theta$
- c. $\frac{1}{16} \int \sin^3 \theta d\theta$
- d. $\frac{1}{64} \int \cos^3 \theta d\theta$
- e. $\frac{1}{16} \int \cos \theta d\theta$ correct choice

Simply identify the integral after the substitution.

Solution:

$x = 4 \tan \theta$
$dx = 4 \sec^2 \theta d\theta$

$$I = \int \frac{4 \sec^2 \theta d\theta}{(16 \tan^2 \theta + 16)^{3/2}} = \int \frac{4 \sec^2 \theta d\theta}{64 (\sec^2 \theta)^{3/2}} = \int \frac{d\theta}{16 \sec \theta}$$

$$= \frac{1}{16} \int \cos \theta d\theta$$

6.

Partial Fractions: Finding Coefficients

New Function Include Completing the Square

Goal: Find the coefficients in the partial fraction expansion:

$$\frac{-2x^2 - x + 2}{x^2(x-1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x-1}$$

- a. $A_1 = 1 \quad A_2 = 2$
- b. $A_1 = -1 \quad A_2 = -2$ correct choice
- c. $A_1 = 2 \quad A_2 = 1$
- d. $A_1 = -2 \quad A_2 = -1$
- e. $A_1 = -2 \quad A_2 = 1$

Just find A_1 and A_2 .

Solution: Clear the denominator: $-2x^2 - x + 2 = A_1x(x-1) + A_2(x-1) + A_3x^2$ (*)

Plug in $x = 0$: $2 = A_2(-1) \quad A_2 = -2$

Differentiate (*): $-4x - 1 = A_1(2x - 1) + A_2 + A_3 2x$

Plug in $x = 0$: $-1 = A_1(-1) + A_2 = -A_1 - 2 \quad A_1 = -1$

7.

Volume Of Revolution _ □ ×

New Problem or Modify or Make Your Own Problem Quit

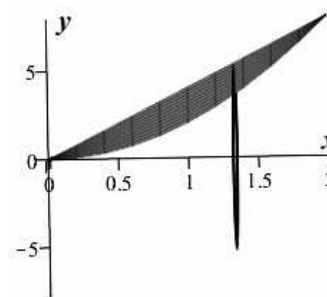
The region above $y = 2x^2$, below $y = 4x$, between $x = 0$ and $x = 2$ is rotated about the x -axis. Find the volume swept out.

- a. $\frac{256}{15}\pi$ correct choice
- b. $\frac{16}{15}\pi$
- c. $\frac{256}{3}\pi$
- d. $\frac{16}{3}\pi$
- e. $\frac{8}{3}\pi$

Solution: The region is shown. It is an x -integral. The vertical slices rotate into washers.

$$V = \int_0^2 \pi(R^2 - r^2) dx = \int_0^2 \pi(16x^2 - 4x^4) dx$$

$$= \pi \left[\frac{16x^3}{3} - \frac{4x^5}{5} \right]_0^2 = \pi \frac{5 \cdot 128 - 3 \cdot 128}{15} = \frac{256}{15}\pi$$



8.

Volume Of Revolution _ □ ×

New Problem or Modify or Make Your Own Problem Quit

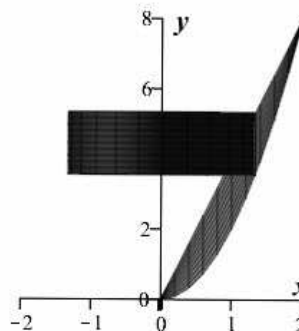
The region above $y = 2x^2$, below $y = 4x$, between $x = 0$ and $x = 2$ is rotated about the y -axis. Find the volume swept out.

- a. $\frac{256}{15}\pi$
- b. $\frac{16}{15}\pi$
- c. $\frac{256}{3}\pi$
- d. $\frac{16}{3}\pi$ correct choice
- e. $\frac{8}{3}\pi$

Solution: The region is shown. It is an x -integral. The vertical slices rotate into cylinders.

$$V = \int_0^2 2\pi rh dx = \int_0^2 2\pi(y)(4x - 2x^2) dx$$

$$= 2\pi \left[\frac{4x^3}{3} - \frac{2x^4}{4} \right]_0^2 = 2\pi \left(\frac{32}{3} - 8 \right) = \frac{16}{3}\pi$$



9.

Surface Area Of Solid Of Revolution

New Problem or Modify or Make Your Own Problem

The curve $x = 2/3 \cdot y^2$, between $y = 0$ and $y = 1$, is rotated about the x -axis. Find the surface area of the surface of revolution.

- a. $\frac{126}{72} \pi$
 b. $\frac{49}{36} \pi$ correct choice
 c. $\frac{49}{144}$
 d. $\frac{49}{72}$
 e. $\frac{49}{36}$

Solution:
$$L = \int_0^1 2\pi r ds = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy = \int_0^1 2\pi y \sqrt{\left(\frac{4}{3}y\right)^2 + 1} dy$$

$$= \int_0^1 \frac{2\pi}{3} y \sqrt{16y^2 + 9} dy = \frac{2\pi}{3} \left[\frac{2(16y^2 + 9)^{3/2}}{3 \cdot 32} \right]_0^1 = \frac{\pi}{72} [(16x^2 + 9)^{3/2}]_0^1$$

$$= \frac{\pi}{72} 25^{3/2} - \frac{\pi}{72} 9^{3/2} = \frac{\pi}{72} (125 - 27) = \frac{49}{36} \pi$$

10.

Work to Lift an Object with a Rope

New Problem

Goal:

Find the work needed to lift a 10 lb object up a 20 ft building using a rope whose density is 5 lb/ft.

- a. 200 ft-lb
 b. 500 ft-lb
 c. 700 ft-lb
 d. 1000 ft-lb
 e. 1200 ft-lb correct choice

Solution: The work to lift just the 10 lb weight is $W_1 = FD = 10 \text{ lb} \cdot 20 \text{ ft} = 200 \text{ ft-lb}$.

Measuring y from the bottom of the building, the work to lift just the rope is

$$W_2 = \int_0^{20} D dF = \int_0^{20} (20 - y) 5 dy = \left[100y - \frac{5}{2}y^2 \right]_0^{20} = 1000 \text{ ft-lb}.$$

So the total work is $W = 200 + 1000 = 1200 \text{ ft-lb}$

11.

Geometric Series

New Series

Goal: Compute the sum of the geometric series (if the sum exists).

$$S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 4 \left(-\frac{2}{5}\right)^n$$

- a. $-\frac{8}{7}$ correct choice
- b. $-\frac{8}{3}$
- c. $\frac{20}{3}$
- d. $\frac{20}{7}$
- e. diverges

Solution: $a = 4\left(-\frac{2}{5}\right) = -\frac{8}{5}$ $r = -\frac{2}{5}$ $S = \frac{a}{1-r} = \frac{-\frac{8}{5}}{1+\frac{2}{5}} = -\frac{8}{7}$

12.

Computing Limits Using Maclaurin Series

New Limit

Goal: Use a Maclaurin Series to evaluate this limit:

$$L = \lim_{x \rightarrow 0} \frac{\sin(x^3) - x^3}{x^9}$$

- a. $-\frac{1}{9!}$
- b. $-\frac{1}{6}$ correct choice
- c. $\frac{1}{6}$
- d. $\frac{1}{9!}$
- e. diverges

Solution: $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} \dots$ $\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \dots$

$$L = \lim_{x \rightarrow 0} \frac{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \dots - x^3}{x^9} = \lim_{x \rightarrow 0} \left(-\frac{1}{3!} + \frac{x^6}{5!} + \dots\right) = -\frac{1}{6}$$

13.

Triangles: Angles in 3D

New Problem Radians

Goal: Find the angle at A in the triangle with vertices:

A = [2, -1, -2] B = [4, -3, -1]

C = [0, -5, 2]

- a. 0
- b. $\frac{\pi}{2}$
- c. $\arccos\left(\frac{2}{3}\right)$
- d. $\arccos\left(\frac{2}{9}\right)$
- e. $\arccos\left(\frac{4}{9}\right)$ correct choice

Solution: $\vec{AB} = (2, -2, 1)$ $\vec{AC} = (-2, -4, 4)$

$$|\vec{AB}| = \sqrt{4+4+1} = 3$$

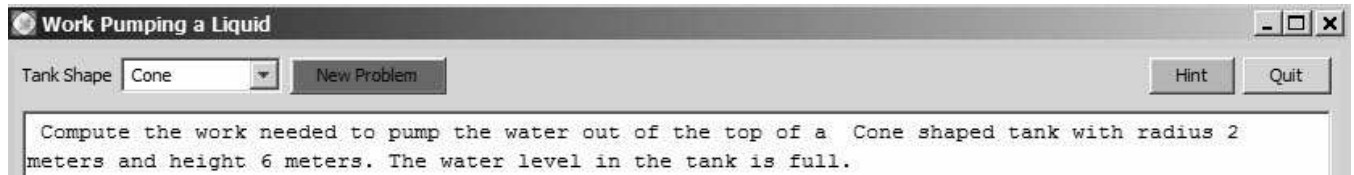
$$|\vec{AC}| = \sqrt{4+16+16} = 6$$

$$\vec{AB} \cdot \vec{AC} = -4 + 8 + 4 = 8$$

$$\cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{8}{3 \cdot 6} = \frac{4}{9} \quad \theta = \arccos\left(\frac{4}{9}\right)$$

Work Out (5 questions, Points indicated. Show all you work.)

14. (20 points)



Write your answer as a multiple of ρg where ρ is the density of water and g is the acceleration of gravity. The vertex of the cone is at the bottom.

Solution: Put $y = 0$ at the bottom of the tank. The slice at height y is lifted a distance $D = 6 - y$.

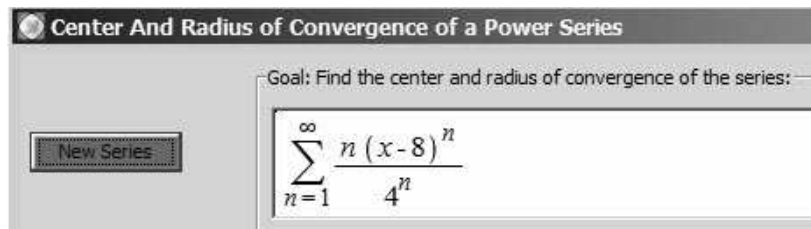
It is a thin disk of radius r satisfying $\frac{r}{y} = \frac{2}{6}$. So $r = \frac{1}{3}y$. The area of the disk is $A = \pi r^2 = \frac{\pi y^2}{9}$.

The volume of the disk is $dV = \frac{\pi y^2}{9} dy$. The weight of the disk is $dF = \rho g dV = \rho g \frac{\pi y^2}{9} dy$.

So the work is

$$\begin{aligned} W &= \int D \cdot dF = \frac{\pi \rho g}{9} \int_0^6 (6 - y)y^2 dy = \frac{\pi \rho g}{9} \left[2y^3 - \frac{y^4}{4} \right]_0^6 = \frac{\pi \rho g}{9} \left(2 \cdot 6^3 - \frac{6^4}{4} \right) \\ &= 24\pi \rho g \left(2 - \frac{6}{4} \right) = 12\pi \rho g \end{aligned}$$

15. (20 points)



Also find the interval of convergence by checking the endpoints.

a. (2 pts) Identify the center: $a = \underline{\quad 8 \quad}$

b. (8 pts) Find the radius of convergence:

Solution: Apply the Ratio Test:

$$\rho = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{(n+1)|x-8|^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n|x-8|^n} = \frac{|x-8|}{4} \lim_{n \rightarrow \infty} \frac{(n+1)}{n} = \frac{|x-8|}{4} < 1$$

$$|x-8| < 4 \quad 4 < x < 12 \quad R = \underline{\quad 4 \quad}$$

c. (8 pts) Check the endpoints:

Solution:

$$x = 4: \quad \sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n n \quad \lim_{n \rightarrow \infty} (-1)^n n = \text{divergent} \neq 0$$

Diverges by the n^{th} -Term Divergence Test

$$x = 12: \quad \sum_{n=1}^{\infty} \frac{n(4)^n}{4^n} = \sum_{n=1}^{\infty} n \quad \lim_{n \rightarrow \infty} n = \infty \neq 0$$

Diverges by the n^{th} -Term Divergence Test

d. (2 pts) Summarize the interval of convergence: $I = \underline{\quad (4, 12) \quad}$

16. (5 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is absolutely convergent, convergent but not absolutely or divergent. Explain all tests you use.

Solution: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent by the Alternating Series Test since the $(-1)^n$ says it is alternating, $\frac{1}{n^{1/3}}$ is decreasing and $\lim_{n \rightarrow \infty} \frac{1}{n^{1/3}} = 0$.

The related absolute series is $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ which is divergent because it is a p -series with

$$p = \frac{1}{3} < 1.$$

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent but not absolutely.

17. (5 points) The series $S = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges by the Integral Test.

If it is approximated by its 100th partial sum S_{100} , compute the integral bound on the error in this approximation.

Solution: The bound is

$$|E_7| = |S - S_{100}| < \int_{100}^{\infty} \frac{1}{n^2 + 1} dn = \left[\arctan(n) \right]_{100}^{\infty} = \frac{\pi}{2} - \arctan(100) \quad (\approx 0.01)$$

18. (5 points) Compute the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 3^{2n+1}}$.

Solution: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ So $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 3^{2n+1}} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$