Name______________________________ Section____________

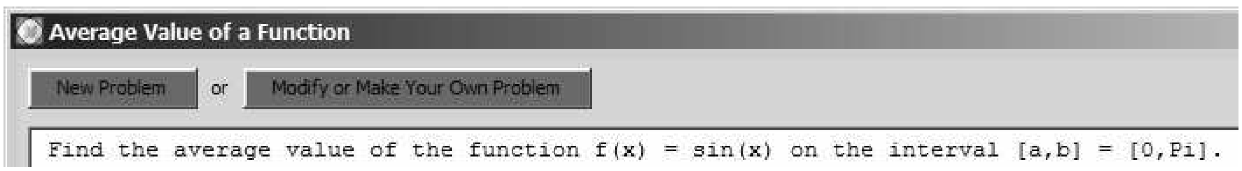
MATH 152 FINAL EXAM Version B Spring 2016

Sections 555-557 Solutions P. Yasskin

1-13	/52
14	/20
15	/20
16	/ 5
17	/ 5
18	/ 5
Total	/107

Multiple Choice: (13 problems, 4 points each)

1.



Average Value of a Function

New Problem or Modify or Make Your Own Problem

Find the average value of the function f(x) = sin(x) on the interval [a,b] = [0,Pi].

a. $\frac{1}{\pi}$

b. $\frac{2}{\pi}$ correct choice

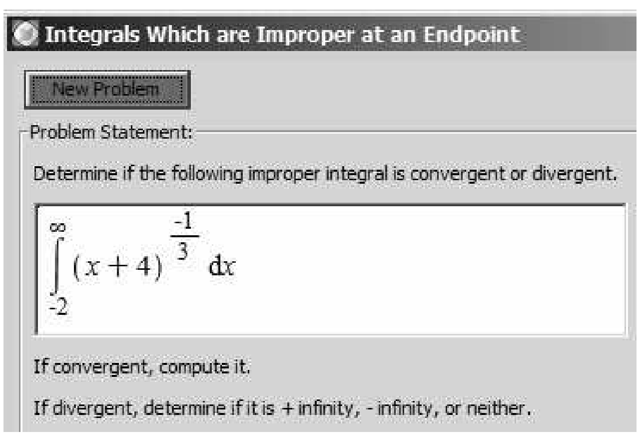
c. 1

d. 2

e. 2π

Solution: $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_0^\pi \sin(x)\,dx = -\frac{1}{\pi}\cos(x)\Big|_0^\pi = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

2.



Integrals Which are Improper at an Endpoint

New Problem

Problem Statement:

Determine if the following improper integral is convergent or divergent.

$$\int_{-2}^{\infty} (x+4)^{\frac{-1}{3}}\,dx$$

If convergent, compute it.

If divergent, determine if it is + infinity, - infinity, or neither.

a. converges to $\frac{3}{2^{1/3}}$

b. converges to $-\frac{3}{2^{1/3}}$

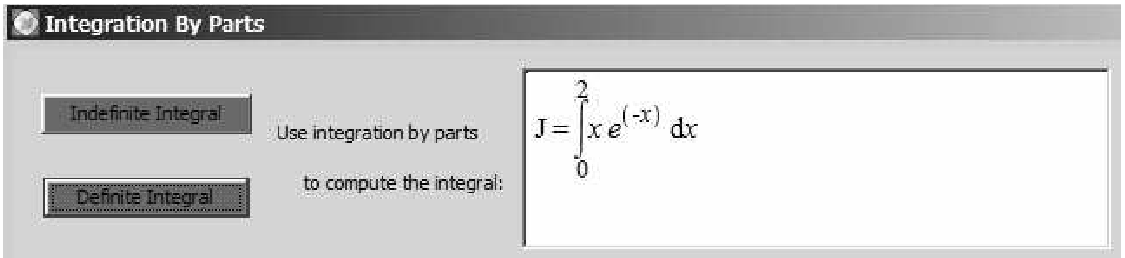
c. diverges to $-\infty$

d. diverges to ∞ correct choice

e. diverges but not to $\pm\infty$

Solution: $\int_{-2}^{\infty}(x+4)^{-1/3}\,dx = \frac{3(x+4)^{2/3}}{2}\Big|_{-2}^{\infty} = \infty - \frac{3(2)^{2/3}}{2} = \infty$

3.



a. $-e^{-2}$

b. $-e^{-2} - 1$

c. $1 - e^{-2}$

d. $1 - 3e^{-2}$ correct choice

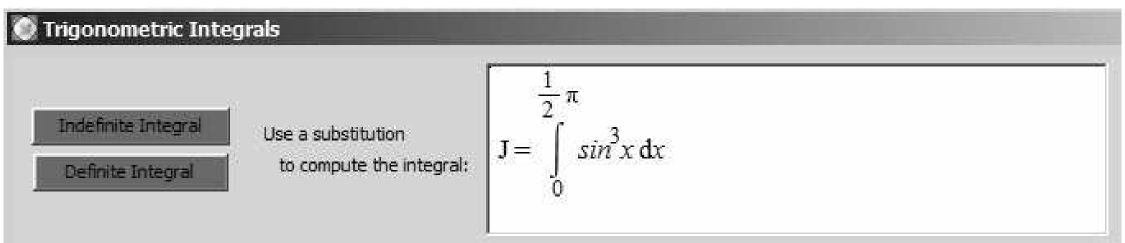
e. $1 + e^{-2}$

Solution:

$u = x$	$dv = e^{-x}dx$
$du = dx$	$v = -e^{-x}$

$$J = \left[-xe^{-x} + \int e^{-x}dx\right]_0^2 = \left[-xe^{-x} - e^{-x}\right]_0^2$$
$$= \left(-2e^{-2} - e^{-2}\right) - \left(-0 - e^{-0}\right) = -3e^{-2} + 1$$

4.



a. $-\frac{4}{3}$

b. $-\frac{1}{4}$

c. $\frac{1}{4}$

d. $\frac{2}{3}$ correct choice

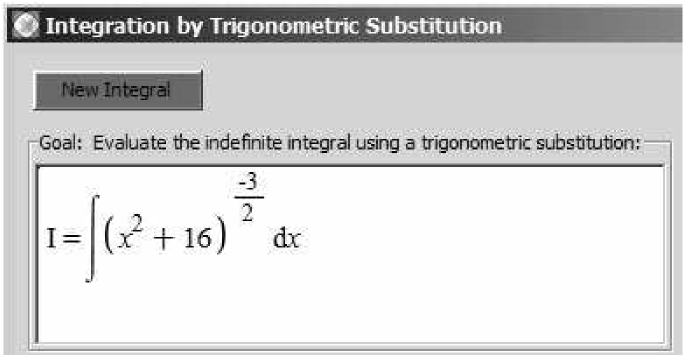
e. $\frac{4}{3}$

Solution:

$u = \cos x$
$du = -\sin x\, dx$

$$J = \int_0^{\pi/2} (1 - \cos^2 x)\sin x\, dx = -\int_1^0 (1 - u^2)\, du = \left[-u + \frac{u^3}{3}\right]_1^0$$
$$= -\left(-1 + \frac{1}{3}\right) = \frac{2}{3}$$

5.



a. $\frac{1}{16} \int \csc^2\theta \, d\theta$

b. $\frac{1}{64} \int \sec^2\theta \, d\theta$

c. $\frac{1}{16} \int \sin^3\theta \, d\theta$

d. $\frac{1}{64} \int \cos^3\theta \, d\theta$

e. $\frac{1}{16} \int \cos\theta \, d\theta$ correct choice

Simply identify the integral after the substitution.

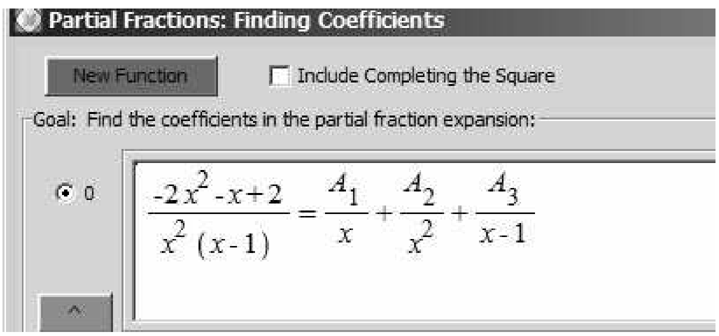
Solution:

$x = 4\tan\theta$
$dx = 4\sec^2\theta \, d\theta$

$$I = \int \frac{4\sec^2\theta \, d\theta}{(16\tan^2\theta + 16)^{3/2}} = \int \frac{4\sec^2\theta \, d\theta}{64(\sec^2\theta)^{3/2}} = \int \frac{d\theta}{16\sec\theta}$$

$$= \frac{1}{16} \int \cos\theta \, d\theta$$

6.



a. $A_1 = 1 \quad A_2 = 2$

b. $A_1 = -1 \quad A_2 = -2$ correct choice

c. $A_1 = 2 \quad A_2 = 1$

d. $A_1 = -2 \quad A_2 = -1$

e. $A_1 = -2 \quad A_2 = 1$

Just find A_1 and A_2.

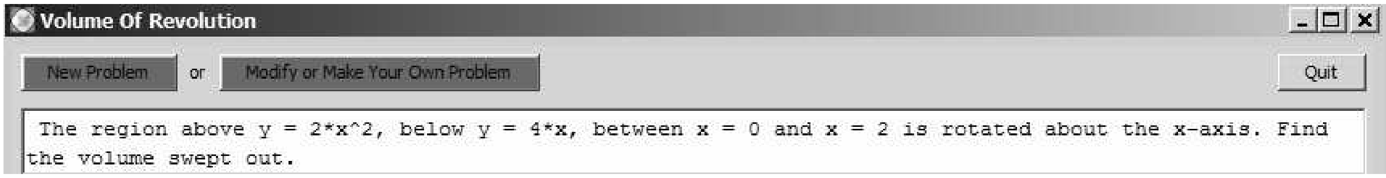
Solution: Clear the denominator: $-2x^2 - x + 2 = A_1 x(x-1) + A_2(x-1) + A_3 x^2$ (*)

Plug in $x = 0$: $2 = A_2(-1) \quad A_2 = -2$

Differentiate (*): $-4x - 1 = A_1(2x-1) + A_2 + A_3 2x$

Plug in $x = 0$: $-1 = A_1(-1) + A_2 = -A_1 - 2 \quad A_1 = -1$

7.



a. $\frac{256}{15}\pi$ correct choice

b. $\frac{16}{15}\pi$

c. $\frac{256}{3}\pi$

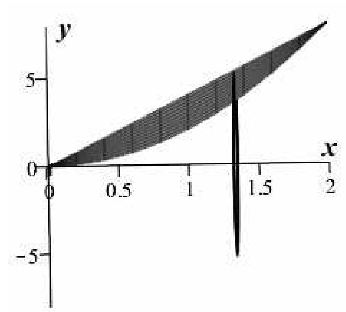
d. $\frac{16}{3}\pi$

e. $\frac{8}{3}\pi$

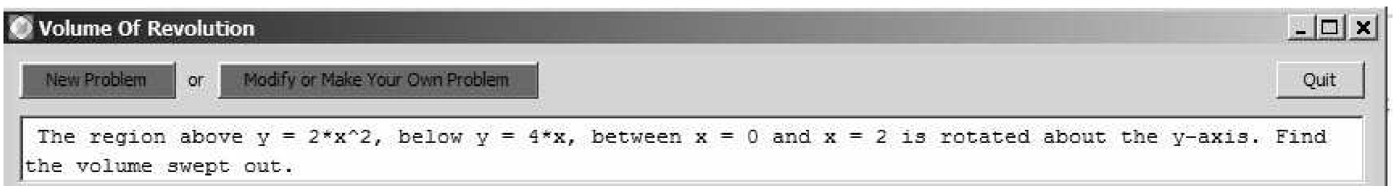
Solution: The region is shown. It is an x-integral. The vertical slices rotate into washers.

$$V = \int_0^2 \pi(R^2 - r^2)\,dx = \int_0^2 \pi(16x^2 - 4x^4)\,dx$$

$$= \pi\left[\frac{16x^3}{3} - \frac{4x^5}{5}\right]_0^2 = \pi\frac{5 \cdot 128 - 3 \cdot 128}{15} = \frac{256}{15}\pi$$



8.



a. $\frac{256}{15}\pi$

b. $\frac{16}{15}\pi$

c. $\frac{256}{3}\pi$

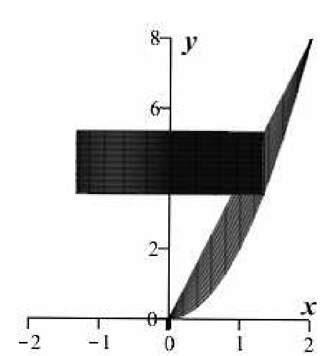
d. $\frac{16}{3}\pi$ correct choice

e. $\frac{8}{3}\pi$

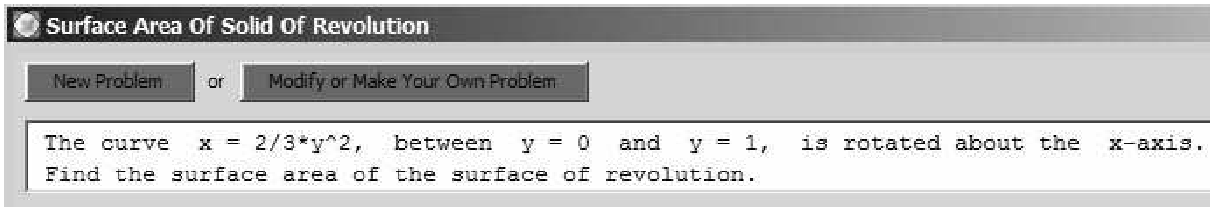
Solution: The region is shown. It is an x-integral. The vertical slices rotate into cylinders.

$$V = \int_0^2 2\pi rh\,dx = \int_0^2 2\pi(y)(4x - 2x^2)\,dx$$

$$= 2\pi\left[\frac{4x^3}{3} - \frac{2x^4}{4}\right]_0^2 = 2\pi\left(\frac{32}{3} - 8\right) = \frac{16}{3}\pi$$



9.



a. $\frac{126}{72}\pi$

b. $\frac{49}{36}\pi$ correct choice

c. $\frac{49}{144}$

d. $\frac{49}{72}$

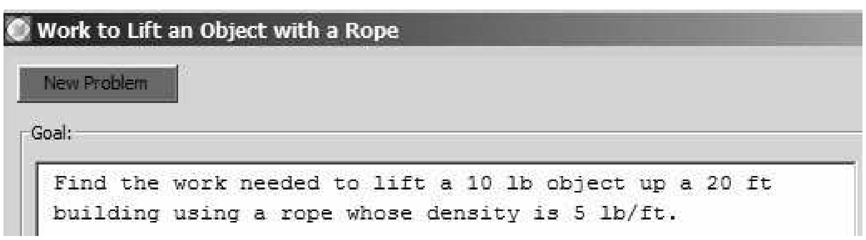
e. $\frac{49}{36}$

Solution: $L = \int_0^1 2\pi r\, ds = \int_0^1 2\pi y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy = \int_0^1 2\pi y\sqrt{\left(\frac{4}{3}y\right)^2 + 1}\; dy$

$= \int_0^1 \frac{2\pi}{3} y\sqrt{16y^2 + 9}\; dy = \frac{2\pi}{3}\left[\frac{2(16y^2+9)^{3/2}}{3 \cdot 32}\right]_0^1 = \frac{\pi}{72}\left[(16x^2+9)^{3/2}\right]_0^1$

$= \frac{\pi}{72}25^{3/2} - \frac{\pi}{72}9^{3/2} = \frac{\pi}{72}(125-27) = \frac{49}{36}\pi$

10.



a. 200 ft-lb

b. 500 ft-lb

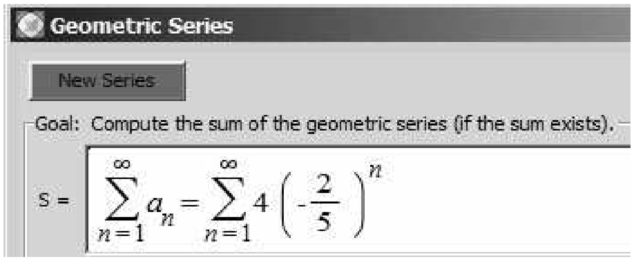
c. 700 ft-lb

d. 1000 ft-lb

e. 1200 ft-lb correct choice

Solution: The work to lift just the 10 lb weight is $W_1 = FD = 10\,\text{lb}\cdot 20\,\text{ft} = 200$ ft-lb.
Measuring y from the bottom of the building, the work to lift just the rope is
$W_2 = \int_0^{20} D\,dF = \int_0^{20}(20-y)\,5\,dy = \left[100y - \frac{5}{2}y^2\right]_0^{20} = 1000$ ft-lb.
So the total work is $W = 200 + 1000 = 1200$ ft-lb

11.



a. $-\frac{8}{7}$ correct choice

b. $-\frac{8}{3}$

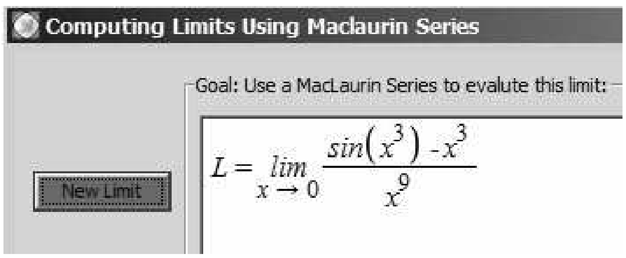
c. $\frac{20}{3}$

d. $\frac{20}{7}$

e. diverges

Solution: $a = 4\left(-\frac{2}{5}\right) = -\frac{8}{5} \qquad r = -\frac{2}{5} \qquad S = \frac{a}{1-r} = \frac{-\frac{8}{5}}{1+\frac{2}{5}} = -\frac{8}{7}$

12.



a. $-\frac{1}{9!}$

b. $-\frac{1}{6}$ correct choice

c. $\frac{1}{6}$

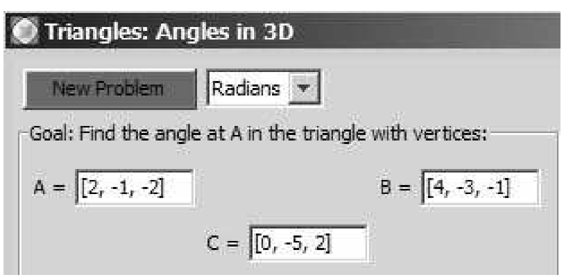
d. $\frac{1}{9!}$

e. diverges

Solution: $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} \cdots \qquad \sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \cdots$

$$L = \lim_{x\to 0} \frac{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \cdots - x^3}{x^9} = \lim_{x\to 0}\left(-\frac{1}{3!} + \frac{x^6}{5!} + \cdots\right) = -\frac{1}{6}$$

13.



a. 0

b. $\frac{\pi}{2}$

c. $\arccos\left(\frac{2}{3}\right)$

d. $\arccos\left(\frac{2}{9}\right)$

e. $\arccos\left(\frac{4}{9}\right)$ correct choice

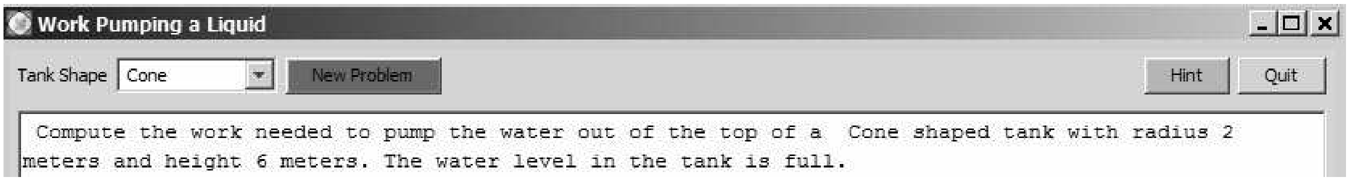
Solution: $\overrightarrow{AB} = (2,-2,1) \qquad \overrightarrow{AC} = (-2,-4,4)$

$\left|\overrightarrow{AB}\right| = \sqrt{4+4+1} = 3 \qquad \left|\overrightarrow{AC}\right| = \sqrt{4+16+16} = 6 \qquad \overrightarrow{AB}\cdot\overrightarrow{AC} = -4+8+4 = 8$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{8}{3\cdot 6} = \frac{4}{9} \qquad \theta = \arccos\left(\frac{4}{9}\right)$$

Work Out (5 questions, Points indicated. Show all you work.)

14. (20 points)



Write your answer as a multiple of ρg where ρ is the density of water and g is the acceleration of gravity. The vertex of the cone is at the bottom.

Solution: Put $y = 0$ at the bottom of the tank. The slice at height y is lifted a distance $D = 6 - y$.

It is a thin disk of radius r satisfying $\frac{r}{y} = \frac{2}{6}$. So $r = \frac{1}{3}y$. The area of the disk is $A = \pi r^2 = \frac{\pi y^2}{9}$.

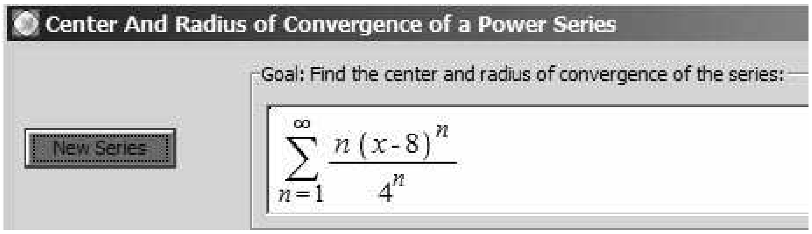
The volume of the disk is $dV = \frac{\pi y^2}{9}\,dy$. The weight of the disk is $dF = \rho g\,dV = \rho g \frac{\pi y^2}{9}\,dy$.

So the work is

$$W = \int D \cdot dF = \frac{\pi \rho g}{9} \int_0^6 (6 - y) y^2\,dy = \frac{\pi \rho g}{9} \left[2y^3 - \frac{y^4}{4} \right]_0^6 = \frac{\pi \rho g}{9} \left(2 \cdot 6^3 - \frac{6^4}{4} \right)$$

$$= 24\pi\rho g \left(2 - \frac{6}{4} \right) = 12\pi\rho g$$

15. (20 points)



Also find the interval of convergence by checking the endpoints.

a. (2 pts) Identify the center: $a = \underline{\quad 8 \quad}$

b. (8 pts) Find the radius of convergence:

Solution: Apply the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)|x-8|^{n+1}}{4^{n+1}} \frac{4^n}{n|x-8|^n} = \frac{|x-8|}{4} \lim_{n\to\infty} \frac{(n+1)}{n} = \frac{|x-8|}{4} < 1$$

$|x-8| < 4 \qquad 4 < x < 12$ $\qquad R = \underline{\quad 4 \quad}$

c. (8 pts) Check the endpoints:

Solution:

$x = 4$: $\displaystyle\sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n n \qquad \lim_{n\to\infty} (-1)^n n =$ divergent $\neq 0$

Diverges by the n^{th}-Term Divergence Test

$x = 12$: $\displaystyle\sum_{n=1}^{\infty} \frac{n(4)^n}{4^n} = \sum_{n=1}^{\infty} n \qquad \lim_{n\to\infty} n = \infty \neq 0$

Diverges by the n^{th}-Term Divergence Test

d. (2 pts) Summarize the interval of convergence: $I = \underline{\quad (4, 12) \quad}$

16. (5 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is absolutely convergent, convergent but not absolutely or divergent. Explain all tests you use.

Solution: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent by the Alternating Series Test since the $(-1)^n$ says it is alternating, $\frac{1}{n^{1/3}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$.

The related absolute series is $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ which is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent but not absolutely.

17. (5 points) The series $S = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the Integral Test.

If it is approximated by its 100^{th} partial sum S_{100}, compute the integral bound on the error in this approximation.

Solution: The bound is

$$|E_7| = |S - S_{100}| < \int_{100}^{\infty} \frac{1}{n^2+1}\, dn = \Big[\arctan(n)\Big]_{100}^{\infty} = \frac{\pi}{2} - \arctan(100) \qquad (\approx 0.01)$$

18. (5 points) Compute the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!3^{2n+1}}$.

Solution: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ So $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!3^{2n+1}} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$