MATH 152
Exam1
Fall 1997
Version A
Solutions

Part I is multiple choice. There is no partial credit. You may not use a calculator.

Part II is work out. Show all your work. Partial credit will be given. You may use your calculator.

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator. You have 1 hour.

1. Compute $\int_0^{\pi/4} \tan x \sec^4 x\, dx$

a. $-\frac{\pi}{2}$

b. $\frac{\pi^2}{32} + \frac{\pi^4}{1024}$

c. $\frac{1}{4}$

d. $\frac{3}{4}$ correctchoice

e. $\frac{5}{6}$

$u = \sec x \qquad du = \sec x \tan x\, dx$

$\int_0^{\pi/4} \tan x \sec^4 x\, dx = \int_{x=0}^{\pi/4} u^3\, du = \left[\frac{u^4}{4} \right] = \left[\frac{\sec^4 x}{4} \right]_0^{\pi/4}$

$= \left[\frac{\sqrt{2}^4}{4} \right] - \left[\frac{1}{4} \right] = 1 - \frac{1}{4} = \frac{3}{4}$ **(d)**

2. The integral $\int_0^{\infty} x^2 e^{-x^3} dx$

a. diverges to $-\infty$

b. converges to $-\frac{1}{3}$

c. converges to 0

d. converges to $\frac{1}{3}$ correctchoice

e. diverges to ∞

$u = x^3 \qquad du = 3x^2\, dx \qquad \frac{1}{3} du = x^2\, dx$

When $x = 0,\ u = 0.$ When $x = \infty,\ u = \infty.$

$\int_0^{\infty} x^2 e^{-x^3} dx = \frac{1}{3} \int_0^{\infty} e^{-u}\, du = \left[-\frac{1}{3} e^{-u} \right]_0^{\infty} = \lim_{u \to \infty} \left(-\frac{1}{3} e^{-u} \right) - \left(-\frac{1}{3} e^0 \right) = 0 + \frac{1}{3} = \frac{1}{3}$ **(d)**

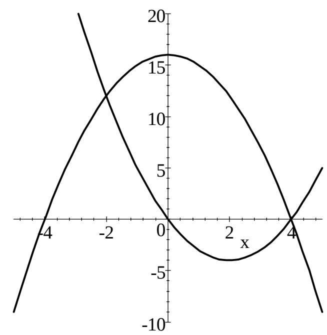
3. At time t in days, a lump of Thorium contains $M = 200e^{(-t/40)}$ kg of radioactive Thorium-234. In other words, the amount of Thorium-234 drops by a factor of $\frac{1}{e}$ every 40 days. Find the average amount of Thorium-234 present in the first 40 days, i.e. between $t = 0$ and $t = 40$.

a. $8000(e-1)$
b. $200\left(1-\frac{1}{e}\right)$ correctchoice
c. $200(e-1)$
d. $4000\left(1+\frac{1}{e}\right)$
e. $100\left(1+\frac{1}{e}\right)$

$$M_{ave} = \frac{1}{40-0}\int_0^{40} 200e^{(-t/40)}\,dt = \frac{200}{40}\left[\frac{e^{(-t/40)}}{\left(\frac{-1}{40}\right)}\right]_0^{40} = -200(e^{-1}-e^0) = 200(1-e^{-1})$$ **(b)**

4. Find the area between the parabolas $y = 16 - x^2$ and $y = x^2 - 4x$.

a. 38
b. 72 correctchoice
c. 96
d. 102
e. 128



$16 - x^2 = x^2 - 4x \qquad 0 = 2x^2 - 4x - 16 \qquad x^2 - 2x - 8 = 0 \qquad (x+2)(x-4) = 0 \qquad x = -2, 4$

$$A = \int_{-2}^{4}(16-x^2)-(x^2-4x)\,dx = \int_{-2}^{4}(16+4x-2x^2)\,dx = \left[16x+2x^2-2\frac{x^3}{3}\right]_{-2}^{4}$$

$$= \left[16\times 4+2\times 4^2-2\frac{4^3}{3}\right]-\left[16(-2)+2(-2)^2-2\frac{(-2)^3}{3}\right]$$

$$= \left[64+32-\frac{128}{3}\right]-\left[-32+8+\frac{16}{3}\right] = 72$$ **(b)**

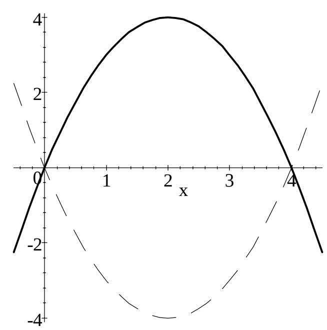
5. Compute $\int_0^{\pi} \sin x\, e^{\cos x}\, dx$

a. $e^{-1} - e$
b. $-e^{-1}$
c. $e^{-1} - 1$
d. $1 - \frac{1}{e}$
e. $e - \frac{1}{e}$ correctchoice

$u = \cos x \qquad du = -\sin x\, dx$
$\int_0^{\pi} \sin x\, e^{\cos x}\, dx = -\int e^u\, du = -e^u = -[e^{\cos x}]_0^{\pi} = -e^{-1} + e^1 = e - \frac{1}{e}$ **(e)**

6. The area below the parabola $y = x(4 - x)$ and above the x-axis is rotated about the x-axis. The volume of the solid swept out is given by

a. $\int_0^4 2\pi y \sqrt{4 - y}\, dy$
b. $\int_0^4 2\pi x^2 (4 - x)\, dx$
c. $2\int_0^2 2\pi x^2 (4 - x)\, dx$
d. $\int_0^4 \pi x^2 (4 - x)^2\, dx$ correctchoice
e. $2\int_0^2 2\pi y \sqrt{4 - y}\, dy$



$V = \int_0^4 \pi y^2\, dx = \int_0^4 \pi [x(4 - x)]^2\, dx = \int_0^4 \pi x^2 (4 - x)^2\, dx$ **(d)**

7. The area below the parabola $y = x(4 - x)$ and above the x-axis is rotated about the y-axis. Find the volume of the solid swept out.

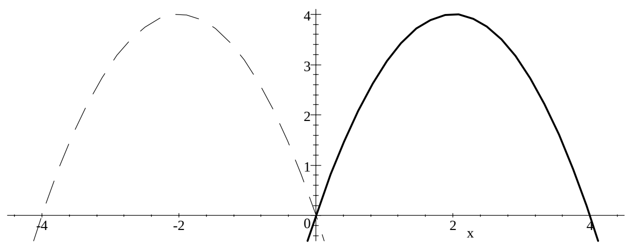
a. $\frac{128}{3}\pi$ correctchoice

b. $\frac{64}{3}\pi$

c. $\frac{512}{15}\pi$

d. $\frac{496}{15}\pi$

e. $\frac{108}{5}\pi$



$$V = \int_0^4 2\pi x h\,dx = \int_0^4 2\pi x[x(4 - x)]\,dx = \int_0^4 2\pi[4x^2 - x^3]\,dx = 2\pi\left[4\frac{x^3}{3} - \frac{x^4}{4}\right]_0^4$$
$$= 512\pi\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{512}{12}\pi = \frac{128}{3}\pi$$ **(a)**

8. Identify which term in the following partial fraction expansion does NOT have the correct form:

$$\frac{2x + 5}{(x - 1)^2(x + 2)(x^2 + 7)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 2} + \frac{D}{x^2 + 7}$$

where A, B, C and D are constants.

a. $\frac{A}{x - 1}$

b. $\frac{B}{(x - 1)^2}$

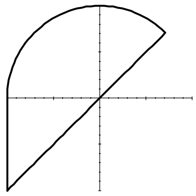
c. $\frac{C}{x + 2}$

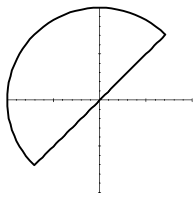
d. $\frac{D}{x^2 + 7}$ correctchoice

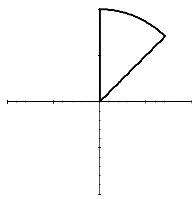
e. None, they all have the correct form.

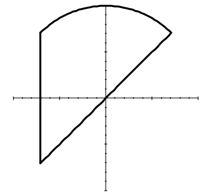
The quadratic term should have a linear numerator: $\frac{Dx + E}{x^2 + 7}$ **(d)**

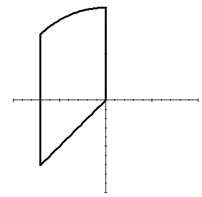
9. The integral $\int_{-2}^{\sqrt{2}} (\sqrt{4-x^2} - x)\,dx$ gives the area of which of the following regions?

a.  correctchoice

b. 

c. 

d. 

e. 

The top is part of the semicircle $y = \sqrt{4-x^2}$. The bottom is part of the line $y = x$.
The integral ranges from $x = -2$ where the semicircle meets the x-axis to $x = \sqrt{2}$ where the semicircle meets the line $y = x$.
Note the line must also extend to $x = -2$. (**a**)

10. Compute $\int_0^1 x^2 e^x\,dx$

- **a**. e
- **b**. $e - 1$
- **c**. $e - 2$ correctchoice
- **d**. $2e - 1$
- **e**. $2e - 2$

Integrate by parts twice: First $u = x^2$ and $dv = e^x dx$. So $du = 2x\,dx$ and $v = e^x$.
Thus the indefinite integral is
$$I = \int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx$$
Second $u = 2x$ and $dv = e^x dx$. So $du = 2\,dx$ and $v = e^x$. Thus
$$I = x^2 e^x - \left[2xe^x - \int 2e^x\,dx\right] = x^2 e^x - 2xe^x + 2e^x$$
So the definite integral is
$$\int_0^1 x^2 e^x\,dx = [x^2 e^x - 2xe^x + 2e^x]_0^1 = [e^1 - 2e^1 + 2e^1] - [0 + 2e^0]$$
$$= e - 2e + 2e - 2 = e - 2$$ (**c**)

Part II: Work Out (10 points each)

Show all your work. Partial credit will be given.
You may use your calculator but only after 1 hour.

11. Compute $\int_{-2}^{2} \sqrt{4-x^2}\, dx$

Substitute $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$:

$$I = \int_{-2}^{2} \sqrt{4-x^2}\, dx = \int \sqrt{4-4\sin^2\theta}\, 2\cos\theta\, d\theta = 4\int \cos^2\theta\, d\theta = 4\int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= 2\left[\theta + \frac{\sin(2\theta)}{2}\right] = 2[\theta + \sin(\theta)\cos(\theta)]$$

Now $x = 2\sin\theta$. So $\sin\theta = \frac{x}{2}$ $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{x^2}{4}}$ and

$\theta = \arcsin\left(\frac{x}{2}\right)$ Thus:

$$I = 2[\theta + \sin(\theta)\cos(\theta)] = 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{1-\frac{x^2}{4}}\right]_{-2}^{2}$$

$$= 2[\arcsin(1)] - 2[\arcsin(-1)] = 4\frac{\pi}{2} = 2\pi$$

12. Compute $\int \frac{4}{(x-1)^2(x+1)}\, dx$

We find the partial fraction expansion. The general form is:

$$\frac{4}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

We clear the denominator:

$$4 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Then we plug in three numbers:

$x = 1$:	$4 = B(2)$	$B = 2$
$x = -1$:	$4 = C(-2)^2$	$C = 1$
$x = 0$:	$4 = A(-1) + B + C$	$A = -1$

So the integral is

$$\int \frac{4}{(x-1)^2(x+1)}\, dx = \int \frac{-1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{x+1}\, dx$$

$$= -\ln|x-1| + \frac{-2}{x-1} + \ln|x+1| + K$$

13. Compute $\int x^5 \sin(x^3 - 1)\,dx$

We first substitute $t = x^3 - 1$. So $dt = 3x^2 dx$ and $x^3 = t + 1$. Thus,

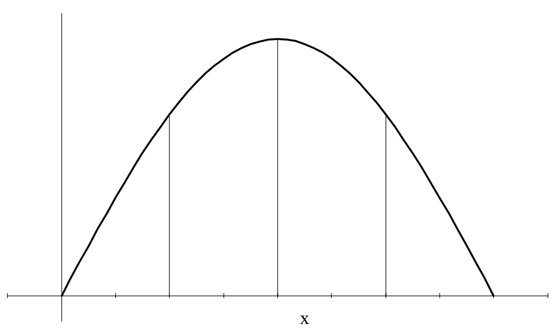
$$I = \int x^5 \sin(x^3 - 1)\,dx = \frac{1}{3}\int (t+1)\sin(t)\,dt$$

We now integrate by parts with $u = t + 1$ and $dv = \sin(t)dt$. So $du = dt$ and $v = -\cos(t)$ Thus,

$$I = \frac{1}{3}\left[-(t+1)\cos(t) + \int \cos(t)dt\right] = \frac{1}{3}[-(t+1)\cos(t) + \sin(t)] + C$$

$$= -\frac{1}{3}x^3\cos(x^3 - 1) + \frac{1}{3}\sin(x^3 - 1) + C$$

14. It is easy to compute $\int_0^\pi \sin x\,dx = 2$ exactly. However, find the approximate value for $\int_0^\pi \sin x\,dx$ using the midpoint rule in a Riemann sum with 4 intervals. Use your calculator to give a decimal value.



The area is divided into 4 regions by the vertical lines at $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$.

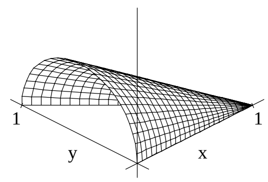
The width of each regions is $\Delta x = \frac{\pi}{4}$ and the midpoints are at $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, and $\frac{7\pi}{8}$.

So the midpoint rule gives

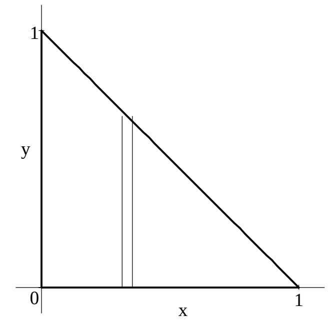
$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{4}\left[\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right)\right]$$

$$\approx .785[.383 + .923 + .923 + .383] \approx 2.052$$

15. The base of a solid is the triangle with corners $(0,0)$, $(0,1)$ and $(1,0)$. The cross-sections perpendicular to the x-axis are semicircles. Compute the volume of the solid.



At the right is a top view:



The slice at x has length $y = 1 - x$. So the semicircles have radius $r = \frac{1-x}{2}$.

So the area of the semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{1-x}{2}\right)^2 = \frac{\pi}{8}(1 - 2x + x^2)$.

So the volume is $V = \int_0^1 A(x)\,dx = \frac{\pi}{8}\int_0^1 (1 - 2x + x^2)\,dx = \frac{\pi}{8}\left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{24}$.