MATH 152

Exam 2

Fall 1997

Version A

Solutions

Part I is multiple choice. There is no partial credit. You may not use a calculator.

Part II is work out. Show all your work. Partial credit will be given. You may use your calculator.

Part I: Multiple Choice (5 points each)
There is no partial credit. You may not use a calculator. You have 1 hour.

1. The natural length of a certain spring is at $2\,\mathrm{m}$. A force of $6\,\mathrm{N}$ will stretch the spring from $2\,\mathrm{m}$ to $4\,\mathrm{m}$. How much work will it take to stretch the spring from $4\,\mathrm{m}$ to $6\,\mathrm{m}$?

a. $18\,\mathrm{J}$ correctchoice

b. $24\,\mathrm{J}$

c. $36\,\mathrm{J}$

d. $48\,\mathrm{J}$

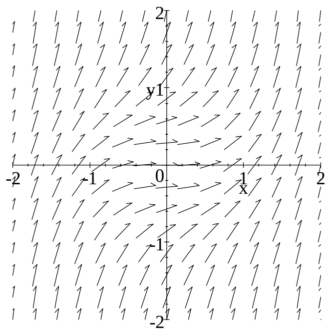
e. $60\,\mathrm{J}$

$F = k(x - x_0) \qquad 6 = k(4-2) \qquad k = 3 \qquad F = 3(x-2)$

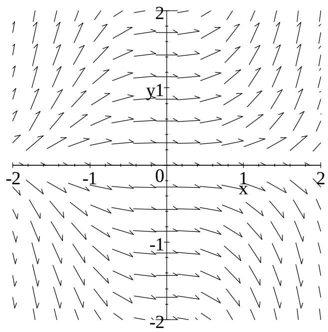
$W = \int_4^6 F\,dx = \int_4^6 3(x-2)\,dx = \left[\frac{3}{2}(x-2)^2\right]_4^6 = \frac{3}{2}(4^2 - 2^2) = 18$ **(a)**

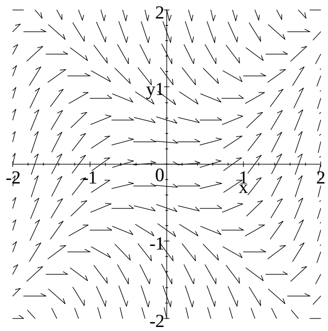
2. The differential equation $\frac{dy}{dx} = x^2 - y^2$ has direction field

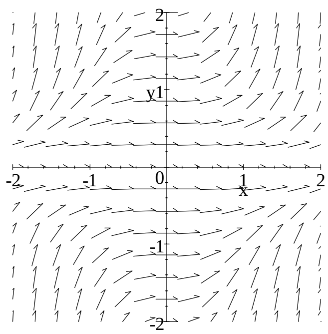
a.

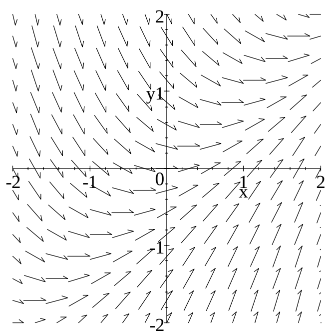


b.



c.  correctchoice

d. 

e. 

The slope must be zero when $x^2 - y^2 = 0$ i.e. along the $45°$ lines $y = \pm x$.
This is only satisfied by **(c)**.

3. The differential equation $\frac{dy}{dx} = (2x+1)(y+1)$ is

a. neither separable nor linear
b. separable but not linear
c. linear but not separable
d. separable and linear correctchoice
e. none of these

The equation is already separated.
To see that it is linear, we write it as $\frac{dy}{dx} = (2x+1)y + (2x+1)$. **(d)**

4. Solve the differential equation $\frac{dy}{dx} = \frac{3}{4}\frac{x^2}{y^3}$ with the initial condition $y(0) = 2$.

a. $x^{3/4} + 2$

b. $\sqrt[3]{x^4 + 8}$

c. $x^{4/3} + 2$

d. $2x^{4/3}$

e. $\sqrt[4]{x^3 + 16}$ correctchoice

Separate: $\int 4y^3 dy = \int 3x^2 dx$ Integrate: $y^4 = x^3 + C$

Use the initial condition: $2^4 = 0^3 + C$ So: $C = 16$

Substitute back: $y^4 = x^3 + 16$ and solve: $y = \sqrt[4]{x^3 + 16}$ (**e**)

5. If **a** $= \langle 4,2,1\rangle$ and **b** $= \langle 3,2,3\rangle$ then **a** $+ 2$**b** $=$

a. $\langle -2,-2,-5\rangle$

b. $\langle 10,6,7\rangle$ correctchoice

c. $\langle 7,4,4\rangle$

d. $\langle 1,0,-2\rangle$

e. $\langle 11,6,5\rangle$

a $+ 2$**b** $= \langle 4,2,1\rangle + \langle 6,4,6\rangle = \langle 10,6,7\rangle$ (**b**)

6. The triangle with vertices $P = (4,2,1)$, $Q = (3,3,1)$ and $R = (3,2,3)$ is

a. equilateral

b. isosceles but not right correctchoice

c. right but not isosceles

d. isosceles and right

e. none of these

$\overrightarrow{PQ} = Q - P = (-1,1,0)$ $\left|\overrightarrow{PQ}\right| = \sqrt{2}$

$\overrightarrow{PR} = R - P = (-1,0,2)$ $\left|\overrightarrow{PR}\right| = \sqrt{5}$

$\overrightarrow{QR} = R - Q = (0,-1,2)$ $\left|\overrightarrow{QR}\right| = \sqrt{5}$ (**b**)

7. In 3-dimensional space the equation $x^2 + z^2 < 16$ describes

a. the interior of a circle of radius 4 in the xz-plane centered at the origin.

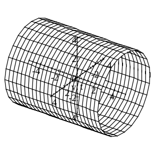
b. the region inside a paraboloid centered along the positive y-axis.

c. the region inside a cone centered along the positive y-axis.

d. the part of the xz-plane outside of a circle of radius 4 centered at the origin.

e. the interior of a cylinder of radius 4 centered on the y-axis. correctchoice

(**e**)



8. The arclength of the curve $y = e^{2x}$ between $x = 1$ and $x = 2$ is given by

a. $\int_1^2 \sqrt{1 + e^{4x}}\, dx$

b. $\int_{e^2}^{e^4} \sqrt{1 + 2e^{2x}}\, dx$

c. $\int_1^2 \sqrt{1 + 4e^{4x}}\, dx$ correctchoice

d. $\int_{e^2}^{e^4} 2\pi \sqrt{1 + e^{2x}}\, dx$

e. $\int_1^2 2\pi x \sqrt{1 + e^{4x}}\, dx$

$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + (2e^{2x})^2}\, dx = \int_1^2 \sqrt{1 + 4e^{4x}}\, dx$ **(c)**

9. The curve $y = \sin(x)$, between $x = 1$ and $x = \frac{\pi}{2}$, is rotated about the x-axis. The surface area of the resulting surface is given by.

a. $\int_1^{\pi/2} 2\pi \sin(x) \sqrt{1 + \cos^2(x)}\; dx$ correctchoice

b. $\int_1^{\pi/2} 2\pi x \sqrt{1 + \cos^2(x)}\; dx$

c. $\int_1^{\pi/2} 2\pi \cos(x) \sqrt{1 + \sin^2(x)}\; dx$

d. $\int_0^1 2\pi y \sqrt{\cos^{-2}(y) + 1}\; dy$

e. $\int_1^{\pi/2} 2\pi x \sqrt{1 + \sin^2(x)}\; dx$

$SA = \int_1^{\pi/2} 2\pi(\text{radius})ds = \int_1^{\pi/2} 2\pi(y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^{\pi/2} 2\pi \sin(x) \sqrt{1 + \cos^2(x)}\; dx$ **(a)**

10. Find a unit vector in the direction of the vector $(3, -4, 12)$

a. $\left(\frac{12}{13}, -\frac{4}{13}, \frac{3}{13}\right)$

b. $\left(-\frac{12}{13}, \frac{4}{13}, -\frac{3}{13}\right)$

c. $\left(-\frac{3}{13}, \frac{4}{13}, -\frac{12}{13}\right)$ correctchoice

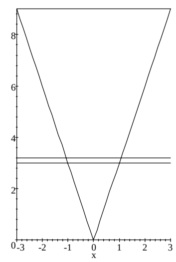
d. $\left(\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)$

e. $\left(-\frac{4}{13}, \frac{12}{13}, \frac{4}{13}\right)$

$\vec{v} = (3, -4, 12) \qquad |\vec{v}| = \sqrt{9 + 16 + 144} = 13 \qquad \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \left(\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)$ **(c)**

Part II: Work Out (10 points each)
Show all your work. Partial credit will be given.
You may use your calculator but only after 1 hour.

11. A plate in the shape of an isosceles triangle with base of $6\,\mathrm{m}$ and altitude of $9\,\mathrm{m}$ is put in water with the base at the surface of the water and the point straight down. Find the total force on the plate. Note the density of water is $1000\,\mathrm{kg/m^3}$ and the acceleration of gravity is $9.8\,\mathrm{m/s^2}$.



The slice is at height y and has length L within the triangle.

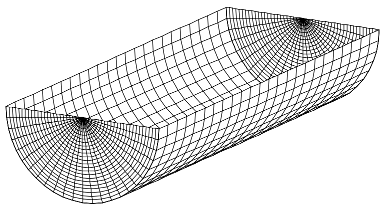
By similar triangles: $\frac{L}{y} = \frac{6}{9}$. So $L = \frac{2}{3}y$.
So the Area of the slice is $dA = L\,dy = \frac{2}{3}y\,dy$.
The depth of the slice below the surface of the water is $h = 9 - y$. So the force is
$F = \int \rho g h\,dA = \int_0^9 1000 \times 9.8 \times (9-y)\frac{2}{3}y\,dy = 1000 \times 9.8 \times \frac{2}{3}\int_0^9 (9y - y^2)\,dy$

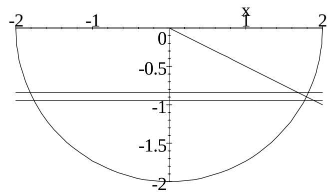
$$= 1000 \times 9.8 \times \frac{2}{3}\left[9\frac{y^2}{2} - \frac{y^3}{3}\right]_0^9 = 1000 \times 9.8 \times \frac{2}{3} \times 9^3 \times \left(\frac{1}{2} - \frac{1}{3}\right) = 793800\,\mathrm{N}$$

12. Solve the differential equation $\frac{dy}{dx} = 3x^2y + 6x^2e^{x^3}$ with the initial condition $y(0) = 5$.

We write the equation in standard form: $\frac{dy}{dx} - 3x^2y = 6x^2e^{x^3}$
We identify the coefficients: $P = -3x$^2 and $Q = 6x^2e^{x^3}$
The integrating factor is $I = e^{\int P dx} = e^{-\int 3x^2 dx} = e^{-x^3}$
Multiply the equation by the integrating factor: $e^{-x^3}\frac{dy}{dx} - 3x^2e^{-x^3}y = 6x^2$
Identify the left as a total derivative: $\frac{d}{dx}\left[e^{-x^3}y\right] = 6x^2$
Integrate both sides: $e^{-x^3}y = \int 6x^2 dx = 2x^3 + C$
Use the initial condition $y(0) = 5$ to get $e^0 5 = 0 + C$ so that $C = 5$
Substitute back to get $e^{-x^3}y = 2x^3 + 5$ or $y = (2x^3 + 5)e^{x^3}$

13. A water trough has the shape of half of a cylinder on its side, as shown in the figure. The radius is $2\,\mathrm{m}$ and the length is $10\,\mathrm{m}$. How much work does it take to pump the water out of the top of the trough if the trough is full of water? The density of water is $1000\,\mathrm{kg/m^3}$.

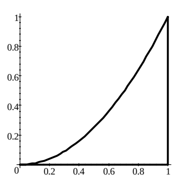




The slice is at depth $h = y$ below the surface of the water.
The length of the slice is $L = 2x = 2\sqrt{4-y^2}$
So its volume is $dV = 10L\,dy = 20\sqrt{4-y^2}\,dy$
and the weight of the slice is $dF = \rho g\,dV = 20\rho g\sqrt{4-y^2}\,dy$ So the work is

$$W = \int_0^2 h\,dF = \int_0^2 20\rho g y\sqrt{4-y^2}\,dy = -10\rho g\int\sqrt{u}\,du = -10\rho g\frac{2u^{3/2}}{3}$$
$$= -10\rho g\frac{2}{3}\Big[(4-y^2)^{3/2}\Big]_0^2 = \frac{160}{3}\rho g = \frac{160}{3}1000\times 9.8 \approx 522667\,\mathrm{J}$$

14. Find the x-coordinate of the centroid of the area bounded by the parabola $y = x^2$ and the lines $x = 1$ and $y = 0$.



The area is $A = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

The moment is $M_y = \int_0^1 x x^2 dx = \int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

So the x-coordinate of the centroid is $\bar{x} = \frac{M_y}{A} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$

15. A barrel initially contains 3 cups of sugar dissolved in 4 gallons of water. You then add pure water at the rate of 2 gallons per minute while the mixture is draining out of a hole in the bottom at 2 gallons per minute. Find the amount of sugar in the barrel after 1 minute.

Note: You must explicitly show the differential equation, the initial condition, the method of solution, the general solution, the solution satisfying the initial condition and the final answer.

Let $S(t)$ be the cups of sugar in the barrel at time t.
So the initial condition is $S(0) = 3$: and the differential equation is:

$$\frac{dS}{dt} = \underbrace{\frac{0\text{cups}}{\text{gal}} \times \frac{2\,\text{gal}}{\text{mn}}}_{IN} - \underbrace{\frac{S\text{cups}}{4\,\text{gal}} \times \frac{2\,\text{gal}}{\text{mn}}}_{OUT} = -\frac{S}{2}$$

We solve: $\int \frac{dS}{S} = -\int \frac{dt}{2}$ $\quad \ln|S| = -\frac{1}{2}t + C$ $\quad |S| = e^{-\frac{1}{2}t+C}$ $\quad S = \pm e^C e^{-\frac{1}{2}t} = Ae^{-\frac{1}{2}t}$
We use the initial condition: $S(0) = 3 = Ae^0 = A$
So the solution is: $S = 3e^{-\frac{1}{2}t}$
Finally after 1 minute there $S(1) = 3e^{-\frac{1}{2}} \approx 1.82\text{cups}$ of sugar in the barrel.