Name________________________ Sec______ ID______________

MATH 152 Honors   Final Exam   Spring 2006

Sections 201,202   Solutions   P. Yasskin

1-12	/48
13	/15+5
14	/20
15	/20
Total	/108

Multiple Choice: (4 points each)

1. The temperature along a probe which is π m long is given by $T = 75 + \sin x$ for $0 \le x \le \pi$. Find the average temperature of the probe.

a. $75 + \dfrac{2}{\pi}$ CorrectChoice

b. $75 - \dfrac{1}{\pi}$

c. 75

d. $75\pi - 1$

e. $75\pi + 2$

$$T_{\text{ave}} = \frac{1}{\pi}\int_0^\pi (75 + \sin x)\,dx = \frac{1}{\pi}\Big[75x - \cos x\Big]_0^\pi = 75 - \frac{1}{\pi}\cos\pi + \frac{1}{\pi}\cos 0 = 75 + \frac{2}{\pi}$$

2. Compute $\int_0^1 xe^{2x}\,dx$.

a. $\dfrac{1}{2}e^2$

b. $\dfrac{1}{4}e^2$

c. $\dfrac{1}{2}(e^2+1)$

d. $\dfrac{1}{4}(e^2+1)$ CorrectChoice

e. $\dfrac{1}{4}(3e^2+1)$

$u = x \qquad dv = e^{2x}\,dx$

$du = dx \qquad v = \dfrac{1}{2}e^{2x}$

$$\int_0^1 xe^{2x}\,dx = \left[\frac{x}{2}e^{2x} - \frac{1}{2}\int_0^1 e^{2x}\,dx\right]_0^1 = \left[\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right]_0^1 = \left(\frac{1}{2}e^2 - \frac{1}{4}e^2\right) - \left(-\frac{1}{4}\right) = \frac{1}{4}(e^2+1)$$

3. Compute $\int_1^{\sqrt{3}} \frac{2x+2}{x(1+x^2)}\,dx$

a. $\ln\sqrt{3} + \frac{\pi}{12}$

b. $\ln 3$

c. $\ln 3 + \frac{\pi}{6}$

d. $\ln\frac{3}{2}$

e. $\ln\frac{3}{2} + \frac{\pi}{6}$ CorrectChoice

$\frac{2x+2}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$ $\qquad 2x+2 = A(1+x^2)+(Bx+C)(x) = (A+B)x^2+Cx+A$

$A = 2 \qquad B = -2 \qquad C = 2$

$\int_1^{\sqrt{3}} \frac{2x+2}{x(1+x^2)}\,dx = \int_1^{\sqrt{3}} \frac{2}{x} + \frac{-2x+2}{1+x^2}\,dx = \int_1^{\sqrt{3}} \frac{2}{x} - \frac{2x}{1+x^2} + \frac{2}{1+x^2}\,dx$

$= [2\ln|x| - \ln|1+x^2| + 2\arctan x]_1^{\sqrt{3}} = \left(2\ln\sqrt{3} - \ln 4 + 2\arctan\sqrt{3}\right) - (2\ln 1 - \ln 2 + 2\arctan 1)$

$= \ln\frac{3}{2} + \frac{\pi}{6}$

4. If $y = f(x)$ is a solution of the differential equation $\frac{dy}{dx} = x + y^2$ satistfing the initial condition $f(1) = 2$, then $f'(1) =$

a. 1

b. 2

c. 3

d. 4

e. 5 CorrectChoice

As a solution, $y = f(x)$ satisfies $f'(x) = x + y^2 = x + f(x)^2$. So $f'(1) = 1 + f(1)^2 = 1 + 2^2 = 5$

5. Compute $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{16^n (2n)!}$

a. 0

b. $\frac{1}{2}$

c. $\frac{1}{\sqrt{2}}$ CorrectChoice

d. 1

e. -1

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sin x$ So $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{16^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{4}\right)^{2n} = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

6. The region between $y = 3x - x^2$ and the x-axis is rotated about the y-axis. Find the volume of the solid swept out.

a. $\frac{81}{5}\pi$

b. $\frac{81}{10}\pi$

c. $\frac{27}{2}\pi$ CorrectChoice

d. 27π

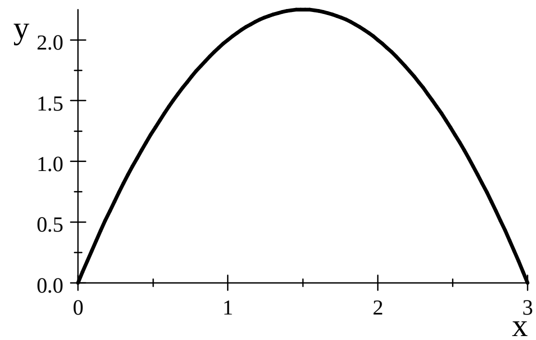
e. $\frac{81}{2}\pi$

x-integral $\quad 3x - x^2 = 0 \quad \Rightarrow \quad x = 0, 3$

y-axis cylinders $\quad r = x \quad h = 3x - x^2$

$$V = \int_0^3 2\pi rh\,dx = \int_0^3 2\pi x(3x - x^2)\,dx$$

$$= 2\pi\left[x^3 - \frac{x^4}{4}\right]_0^3 = 2\pi\left(27 - \frac{81}{4}\right) = \frac{27}{2}\pi$$



7. The region between $y = 3x - x^2$ and the x-axis is rotated about the x-axis. Find the volume of the solid swept out.

a. $\frac{81}{5}\pi$

b. $\frac{81}{10}\pi$ CorrectChoice

c. $\frac{27}{2}\pi$

d. 27π

e. $\frac{81}{2}\pi$

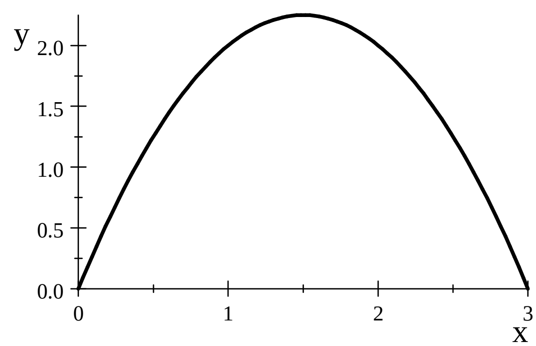
x-integral $\quad 3x - x^2 = 0 \quad \Rightarrow \quad x = 0, 3$

x-axis disk $\quad r = y = 3x - x^2$

$$V = \int_0^3 \pi r^2\,dx = \int_0^3 \pi(3x - x^2)^2\,dx$$

$$= \int_0^3 \pi(x^4 - 6x^3 + 9x^2)\,dx = \pi\left[\frac{x^5}{5} - \frac{3x^4}{2} + 3x^3\right]_0^3$$

$$= \pi\left(\frac{3^5}{5} - \frac{3^5}{2} + 3^4\right) = 81\pi\left(\frac{3}{5} - \frac{3}{2} + 1\right) = \frac{81}{10}\pi$$



8. Find the length of the curve $y = \ln(\cos x)$ for $0 \le x \le \frac{\pi}{4}$.

a. $\ln\left(\sqrt{2}\right)$

b. $\ln\left(\frac{1}{\sqrt{2}}\right)$

c. $\frac{1}{\ln\left(\sqrt{2}\right)}$

d. $\ln\left(\sqrt{2} - 1\right)$

e. $\ln\left(\sqrt{2} + 1\right)$ CorrectChoice

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int ds = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx = \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= \ln\left(\sqrt{2} + 1\right) - \ln(1) = \ln\left(\sqrt{2} + 1\right)$$

9. The curve $x = t^3$, $y = t^2$ for $0 \le t \le 1$ is rotated about the x-axis. Which integral gives the area of the surface swept out?

a. $\int_0^1 \pi t^2 \sqrt{9t^4 + 4t^2}\, dt$

b. $\int_0^1 2\pi t^3 \sqrt{9t^4 + 4t^2}\, dt$

c. $\int_0^1 \pi t^3 \sqrt{9t^2 + 4}\, dt$

d. $\int_0^1 2\pi t^3 \sqrt{9t^2 + 4}\, dt$ CorrectChoice

e. $\int_0^1 2\pi t^2 \sqrt{9t^2 + 4}\, dt$

$$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = 2t \qquad r = y = t^2$$

$$A = \int 2\pi r\, ds = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt = \int_0^1 2\pi t^3 \sqrt{9t^2 + 4}\, dt$$

10. Compute $\int_2^6 \frac{1}{(x-2)^{3/2}}\, dx$

a. -2

b. -1

c. 1

d. 2

e. The integral diverges. CorrectChoice

$$\int_2^6 \frac{1}{(x-2)^{3/2}}\, dx = \int_2^6 (x-2)^{-3/2}\, dx = \Big[-2(x-2)^{-1/2}\Big]_2^6 = -1 + \lim_{x \to 2^+} \frac{2}{(x-2)^{1/2}} = \infty$$

11. Compute $\lim\limits_{x\to 0} \dfrac{\sin(x) - x}{x^2(e^x - e^{-x})}$

a. $-\infty$

b. $\dfrac{-1}{12}$ Correct Choice

c. $\dfrac{1}{6}$

d. $\dfrac{1}{3}$

e. ∞

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^2(e^x - e^{-x})} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x}{x^2\left(\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x^2\left(2x + 2\frac{x^3}{3!} + -\cdots\right)} = \lim_{x\to 0} \frac{-\frac{1}{3!} + \frac{x^2}{5!} - \cdots}{\left(2 + 2\frac{x^2}{3!} + -\cdots\right)} = \frac{-1}{12}$$

12. If you approximate $f(x) = \ln(1 - x)$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ by its 3^{rd} degree Maclaurin polynomial $T_3(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3$, then the Taylor Remainder Theorem says the error is less than

Taylor Remainder Theorem:
If $T_n(x)$ is the n^{th} degree Taylor polynomial for $f(x)$ about $x = a$ then there is a number c between a and x such that the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{(n+1)}$$

a. $\dfrac{1}{4}$ CorrectChoice

b. $\dfrac{1}{8}$

c. $\dfrac{1}{16}$

d. $\dfrac{1}{32}$

e. $\dfrac{1}{64}$

$$f(x) = \ln(1 - x) \qquad f'(x) = \frac{-1}{1 - x} \qquad f''(x) = \frac{-1}{(1 - x)^2} \qquad f'''(x) = \frac{-2}{(1 - x)^3} \qquad f^{(4)}(x) = \frac{-6}{(1 - x)^4}$$

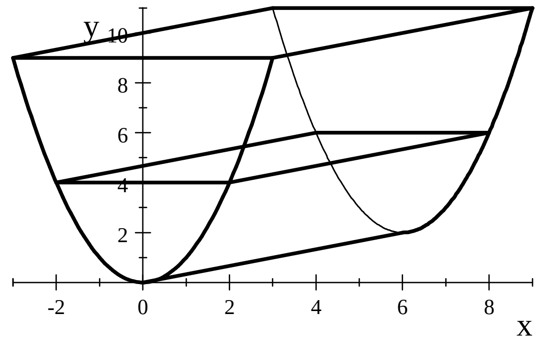
$$M = \max_{x\in[-.5,.5]} |f^{(4)}(x)| = \frac{6}{(1 - .5)^4} = 96$$

For $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $|x| \le \frac{1}{2}$

$$|R_3(x)| = \left|\frac{f^{(4)}(c)}{4!}x^4\right| \le \frac{M}{4!}|x|^4 = \frac{96}{24}\left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

13. (15 points) A water trough is 10 ft long and has vertical ends in the shape of the paraboloid $y = x^2$ up to a height of 9 ft. It is filled with water.

Take the density of water as $\rho g = 60 \dfrac{\text{lb}}{\text{ft}^3}$.



Do one of the following two problems. (**5 points extra credit if you do both**.)

a. Find the work done to pump the water out the top of the trough.

The slice of water at height y with thickness dy has width $2x = 2\sqrt{y}$ and length 10.

So its volume is $dV = 20\sqrt{y}\,dy$ and its weight is $dF = \rho g\,dV = 1200\sqrt{y}\,dy$.

It must be lifted a distance $h = 9 - y$. So the work is

$$W = \int_0^9 h\,dF = \int_0^9 (9-y)1200\sqrt{y}\,dy = 1200\int_0^9 (9y^{1/2} - y^{3/2})\,dy = 1200\left[\frac{9 \cdot 2y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^9$$

$$= 1200\left(3 \cdot 2 \cdot 9^{3/2} - \frac{2 \cdot 9^{5/2}}{5}\right) = 1200\left(3^4 \cdot 2 - \frac{2 \cdot 3^5}{5}\right) = 1200 \cdot 81 \cdot 2 \cdot \frac{2}{5} = 77760$$

b. Find the force of the water on the end of the trough.

The slice of water at height y with thickness dy has width $2x = 2\sqrt{y}$.

So its area on the end is $dA = 2\sqrt{y}\,dy$.

This slice is a distance $h = 9 - y$ below the surface of the water.

So the pressure is $P = \rho g h = 60(9-y)$. So the force is

$$F = \int_0^9 P\,dA = \int_0^9 60(9-y)2\sqrt{y}\,dy = 120\int_0^9 (9y^{1/2} - y^{3/2})\,dy = 120\left[\frac{9 \cdot 2y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^9$$

$$= 120\left(3 \cdot 2 \cdot 9^{3/2} - \frac{2 \cdot 9^{5/2}}{5}\right) = 120\left(3^4 \cdot 2 - \frac{2 \cdot 3^5}{5}\right) = 120 \cdot 81 \cdot 2 \cdot \frac{2}{5} = 7776$$

14. (20 points) Consider the sequence defined recursively by

$$a_1 = 4 \quad \text{and} \quad a_{n+1} = \sqrt{2a_n}$$

a. (4 pt) Write out the first 4 terms and simplify, e.g. $\sqrt{8} = 2\sqrt{2}$

$a_1 =$ 4 $\qquad a_2 =$ $\sqrt{2 \cdot 4} = \sqrt{8} = 2\sqrt{2}$

$a_3 =$ $\sqrt{2 \cdot 2\sqrt{2}} = 2\sqrt[4]{2}$ $\qquad a_4 =$ $\sqrt{2 \cdot 2\sqrt[4]{2}} = 2\sqrt[8]{2}$

b. (5 pt) Is the sequence increasing or decreasing? Prove it.

The sequence is decreasing. We need to prove $a_{n+1} < a_n$.

First note $a_2 = 2\sqrt{2} < 4 = a_1$.

Now assume $a_{k+1} < a_k$. Then $2a_{k+1} < 2a_k$ and $\sqrt{2a_{k+1}} < \sqrt{2a_k}$ or $a_{k+2} < a_{k+1}$.

So $a_{n+1} < a_n$.

c. (5 pt) If it is increasing, find an upper bound.
If it is decreasing, find a lower bound. Prove it.

The sequence is bounded below by 2. We need to prove $a_n > 2$.

First note $a_1 = 4 > 2$.

Now assume $a_k > 2$. Then $a_{k+1} = \sqrt{2a_k} > \sqrt{2 \cdot 2} = 2$.

So $a_n > 2$.

d. (2 pt) What do (b) and (c) imply?

That the sequence has a limit which is ≥ 2.

e. (4 pt) Find the limit of the sequence.

Method 1: Let $L = \lim\limits_{n\to\infty} a_n$. Then $L = \sqrt{2L}$

So $L^2 = 2L$ or $L^2 - 2L = 0$ or $L = 0$ or $L = 2$.

By (b) and (c), $L = 2$.

Method 2: $a_n = 2 \cdot 2^{1/2^{n-1}}$

So $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2 \cdot 2^{1/2^{n-1}} = 2 \cdot 2^{\lim_{n\to\infty} 1/2^{n-1}} = 2 \cdot 2^0 = 2$

15. (20 points) A brine solution that contains 0.4 kg of salt per liter flows at a constant rate of 2 liters per minute into a large tank that initially held 100 liters of brine that contains 0.5 kg of salt per liter. The solution in the tank is kept well mixed and flows out of the tank at 2 liters per minute. Let $S(t)$ be the amount of salt on the tank at time t.

a. (8 pt) Set up the differential equation and initial condition for $S(t)$.

$$\frac{dS}{dt}\frac{\text{kg}}{\text{min}} = \underbrace{0.4\frac{\text{kg}}{\text{L}}2\frac{\text{L}}{\text{min}}}_{\text{in}} - \underbrace{\frac{S(t)\text{kg}}{100\text{L}}2\frac{\text{L}}{\text{min}}}_{\text{out}} \qquad S(0) = 0.5\frac{\text{kg}}{\text{L}}100\text{L} = 50\text{kg}$$

$$\frac{dS}{dt} = 0.8 - .02S(t) \qquad S(0) = 50$$

b. (12 pt) How much salt is in the tank after 50 minutes?

Separable: $\dfrac{dS}{0.8-.02S(t)} = dt \qquad \displaystyle\int \frac{dS}{0.8-.02S(t)} = \int dt \qquad \frac{-1}{.02}\ln|0.8-.02S(t)| = t + C$

$|0.8-.02S(t)| = e^{-.02t-.02C} \qquad 0.8-.02S(t) = Ae^{-.02t}$ where $A = \pm e^{-.02C}$

$.02S(t) = 0.8 - Ae^{-.02t} \qquad S(t) = \dfrac{0.8}{.02} - \dfrac{A}{.02}e^{-.02t} = 40 - 50Ae^{-.02t}$

$S = 50$ when $t = 0$: $\quad 50 = 40 - 50A \qquad 50A = -10 \qquad A = -.2$

$S(t) = 40 + 10e^{-.02t}$

$S(50) = 40 + 10e^{-.02 \cdot 50} = 40 + \dfrac{10}{e} \approx 43.679$

OR

Linear: $\dfrac{dS}{dt} + .02S(t) = 0.8 \qquad I = e^{\int .02\,dt} = e^{.02t}$

$e^{.02t}\dfrac{dS}{dt} + .02e^{.02t}S(t) = 0.8e^{.02t} \qquad \dfrac{d}{dt}(e^{.02t}S) = 0.8e^{.02t}$

$e^{.02t}S = \displaystyle\int 0.8e^{.02t}\,dt = \frac{0.8}{.02}e^{.02t} + C = 40e^{.02t} + C$

$S = 50$ when $t = 0$: $\quad 50 = 40 + C \qquad C = 10$

$e^{.02t}S = 40e^{.02t} + 10 \qquad S = 40 + 10e^{-.02t}$

$S(50) = 40 + 10e^{-.02 \cdot 50} = 40 + \dfrac{10}{e} \approx 43.679$