Spring 2006 Math 152
Exam 3A: Solutions
Mon, 01/May ©2006, Art Belmonte

Notes

- The notation $b_n \downarrow 0$ means $0 < b_{n+1} \le b_n$ and $\lim b_n = 0$; i.e., the sequence $\{b_n\}$ decreases to zero in the limit.
- GFF ["Generalized Fun Fact"]: Let $p(n) = \sum_{k=0}^{m} c_k n^k$ be a polynomial in n with $c_m > 0$. Then $\lim_{n\to\infty} \sqrt[n]{p(n)} = 1$. This boosts the power of the Root Test, which was optionally covered in some classes. (In most classes, only the Ratio Test was covered.)

1. (b) Algebraic manipulation gives

$$\begin{aligned} &\lim_{n\to\infty}\left(\frac{n^2+1}{n^2}\,\sin\left(\frac{\pi n}{2n+1}\right)\right) \\ =\ &\lim_{n\to\infty}\left(\frac{1+\frac{1}{n^2}}{1}\,\sin\left(\frac{\pi}{2+\frac{1}{n}}\right)\right) \\ =\ &\sin\left(\frac{\pi}{2}\right) = 1. \end{aligned}$$

2. (d) Only the series $\sum \frac{(-1)^n}{\sqrt[4]{n}}$ converges, but not absolutely.

   - I. The series $\sum \frac{(-1)^n}{n^8}$ converges absolutely since $\sum |a_n| = \sum \frac{1}{n^8}$, a convergent p-series ($p = 8 > 1$).
   - II. The series $\sum (-1)^{n+1}\left(\frac{n^5+2}{n^3}\right)$ diverges by the Test for Divergence since $\lim a_n \ne 0$. (Indeed, the limit does not exist since $\limsup a_n = \infty$ and $\liminf a_n = -\infty$.)
   - III. The alternating series $\sum \frac{(-1)^n}{\sqrt[4]{n}}$ converges by the Alternating Series Test since $b_n = |a_n| = \frac{1}{\sqrt[4]{n}} \downarrow 0$. Note, however, that $\sum |a_n| = \sum \frac{1}{n^{1/4}}$ is a divergent p-series ($p = \frac{1}{4} \le 1$). Accordingly, $\sum \frac{(-1)^n}{\sqrt[4]{n}}$ converges, but is not absolutely convergent. (We say it is *conditionally convergent*.)

3. (b) We have $\dfrac{1}{1-(-x^4)} = \sum_{n=0}^{\infty}\left(-x^4\right)^n = \sum_{n=0}^{\infty}(-1)^n x^{4n}$, provided $\left|-x^4\right| < 1$ or $|x| < 1$.

4. (d) The series $\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}$ is a telescoping sum.

   - First do a partial fraction decomposition.

$$\begin{aligned} \frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\ 1 &= A(n+1) + Bn \\ 0n + 1 &= (A+B)\,n + A \end{aligned}$$

   Thus $A + B = 0$ and $A = 1$, whence $B = -A = -1$. Therefore, $\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$.

   - Now look at the sequence of partial sums to determine its limit s, the sum of the series.

$$\begin{aligned} s_1 &= 1 - \tfrac{1}{2} \\ s_2 &= \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) = 1 - \tfrac{1}{3} \\ &\vdots \\ s_n &= 1 - \tfrac{1}{n+1} \\ s = \lim_{n\to\infty} s_n &= 1 \end{aligned}$$

5. (c) Only statement III is true.

   - I. If $\lim b_n = 0$, then $\sum b_n$ converges. *This is false.* A counterexample is the harmonic series $\sum \frac{1}{n}$, a divergent p-series ($p = 1 \le 1$).
   - II. If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. *This is false.* A countexample is $\sum a_n = \frac{1}{n^2}$ and $\sum b_n = \sum \frac{1}{n}$. Note that $\sum \frac{1}{n}$, the harmonic series, diverges. Yet while $0 \le a_n = \frac{1}{n^2} \le \frac{1}{n} = b_n$, we see that $\sum \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$).
   - III. If $\sum a_n$ converges, then $\lim a_n = 0$. *This is true.* As stated in Section 10.2, this condition is necessary for a series to converge.

6. (d) Compute the sum of this geometric series via the Geometric Series Theorem.

$$\sum_{n=1}^{\infty}\frac{2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} = \frac{a}{1-r} = \frac{2/3}{1-\frac{2}{3}} = \frac{2/3}{1/3} = 2$$

7. (d) The series $\sum_{n=3}^{\infty}\dfrac{1}{n\ln n}$ diverges by the Integral Test.

$$\begin{aligned} \int_3^{\infty}\frac{1}{x\ln x}\,dx &= \lim_{t\to\infty}\int_3^t \frac{1}{\ln x}\,\frac{1}{x}\,dx \\ &= \lim_{t\to\infty}\int_{\ln 3}^{\ln t}\frac{1}{u}\,du \quad (\text{sub } u = \ln x) \\ &= \lim_{t\to\infty} \ln u\Big|_{\ln 3}^{\ln t} \\ &= \lim_{t\to\infty}\left(\ln(\ln t) - \ln(\ln 3)\right) = \infty \end{aligned}$$

8. (d) Note that $-\dfrac{1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$. Since $\lim_{n\to\infty}\left(-\dfrac{1}{n}\right) = 0$ and $\lim_{n\to\infty}\dfrac{1}{n} = 0$, we conclude that $\lim_{n\to\infty}\dfrac{\sin n}{n} = 0$ by the Squeeze Theorem.

9. (c) The series $\sum_{n=1}^{\infty}\dfrac{x^{2n}}{\sqrt{n}}$ converges for $-1 < x < 1$ as follows.

- The Ratio Test gives

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{|x|^{2n+2}}{\sqrt{n+1}}\frac{\sqrt{n}}{|x|^{2n}}\right) \\
&= \lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}}\,|x|^2 \\
&= |x|^2 \overset{\text{need}}{<} 1 \\
\implies & |x| < 1.
\end{aligned}$$

  Hence the radius of convergence is $R = 1$.
- [Alternatively, use the Root Test along with the GFF. As $n \to \infty$, we have

$$\sqrt[n]{|a_n|} = \frac{\left(|x|^{2n}\right)^{1/n}}{\sqrt[n]{\sqrt{n}}} = \frac{|x|^2}{\sqrt{\sqrt[n]{n}}} \to |x|^2 \overset{\text{need}}{<} 1,$$

  whence $|x| < 1$ and $R = 1$.]
- At the endpoints $x = \pm 1$, the series is $\sum \frac{1}{n^{1/2}}$, a divergent p-series ($p = \frac{1}{2} \le 1$). So the interval of convergence is $I = (-1, 1)$.

10. (a) The series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{(1+n)\,n^p} = \sum_{n=1}^{\infty}\frac{1}{n^p + n^{p+1}}$$

converges for $p > 0$ by the Limit Comparison Test.

- Let $\sum b_n = \sum \frac{1}{n^{p+1}}$. Then
  $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{p+1}}{(1+n)\,n^p} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1 > 0.$
  Since $\sum \frac{1}{n^{p+1}}$ converges only if $q = p + 1 > 1$ or $p > 0$, we conclude by the Limit Comparison Test that $\sum \frac{1}{(1+n)\,n^p}$ converges only if $p > 0$. [Brian Winn]
- Here is an alternative proof that employs repeated use of the Comparison Test instead.
  - First, $\sum_{n=1}^{\infty}\frac{1}{n^p + n^{p+1}} \le \sum_{n=1}^{\infty}\frac{1}{n^{p+1}} = \sum_{n=1}^{\infty}\frac{1}{n^q}$, which converges provided $p + 1 = q > 1$ or $p > 0$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{(1+n)\,n^p}$ converges for $p > 0$ by the Comparison Test.
  - Next,

$$\sum_{n=1}^{\infty}\frac{1}{n^p + n^{p+1}} \ge \sum_{n=1}^{\infty}\frac{1}{n^{p+1} + n^{p+1}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{p+1}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^q},$$

    which diverges for $p + 1 = q \le 1$ or $p \le 0$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{(1+n)\,n^p}$ diverges for $p \le 0$ by the Comparison Test.
  - Accordingly, $\sum_{n=1}^{\infty}\frac{1}{(1+n)\,n^p}$ converges *only* when $p > 0$.

11. The power series $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\,4^n}$ has radius of convergence $R = 4$ and interval of convergence $I = [-1, 7)$.

- The Ratio Test gives

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{|x-3|^{n+1}}{(n+1)\,4^{n+1}}\frac{n4^n}{|x-3|^n}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)\frac{|x-3|}{4} \\
&= \frac{|x-3|}{4} \overset{\text{need}}{<} 1 \\
\implies & |x-3| < 4.
\end{aligned}$$

  Hence the radius of convergence is $R = 4$.
- [Alternatively, use the Root Test along with the GFF. As $n \to \infty$, we have

$$\sqrt[n]{|a_n|} = \frac{|x-3|}{4\sqrt[n]{n}} = \frac{|x-3|}{4} \overset{\text{need}}{<} 1,$$

  whence $|x-3| < 4$ and $R = 4$.]
- At the left endpoint $x = -1$, we have the alternating series $\sum \frac{(-1)^n}{n}$, which converges by the Alternating Series Test since $b_n = |a_n| = \frac{1}{n} \downarrow 0$.
- At the right endpoint $x = 7$, we have the harmonic series $\sum \frac{1}{n}$, a divergent p-series ($p = 1 \le 1$).
- So the interval of convergence is $I = [-1, 7)$.

12. (a) The series $\sum_{n=1}^{\infty}\frac{e^n}{n!}$ converges absolutely (and hence converges) by the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n}\right) \\
&= \lim_{n\to\infty}\frac{e}{n+1} = 0 < 1
\end{aligned}$$

(b) The alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\ln n}$ converges by the Alternating Series Test since $b_n = \left|\frac{1}{a_n}\right| = \frac{1}{1+\ln n} \downarrow 0$.

(c) The series $\sum_{n=1}^{\infty}\frac{n}{n^{2.001}+4}$ converges by the Comparison Theorem since

$$0 \le \frac{n}{n^{2.001}+4} \le \frac{n}{n^{2.001}} = \frac{1}{n^{1.001}}$$

and $\sum \frac{1}{n^{1.001}}$ is a convergent p-series ($p = 1.001 > 1$).

13. Let $f(x) = e^{3x}$ and $a = 2$. Then $f^{(n)}(x) = 3^n e^{3x}$ and $f^{(n)}(2) = 3^n e^6$. So the Taylor series for $f(x) = e^{3x}$ at $a = 2$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty}\frac{3^n e^6}{n!}(x-2)^n.$$

14. (a) Recall that $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n+1}}{(2n+1)!}$ for $z \in \mathbb{R}$. Thus

$$\begin{aligned}
\int_0^1 \sin\left(x^2\right)\, dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^2\right)^{2n+1}}{(2n+1)!}\, dx \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+2}}{(2n+1)!}\, dx \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+3}}{(2n+1)!\,(4n+3)} \right) \Bigg|_{x=0}^{x=1} \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n\, 1^{4n+3}}{(2n+1)!\,(4n+3)} \right) - \left(\sum_{n=0}^{\infty} 0 \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)}
\end{aligned}$$

(b) The Alternating Series Estimation Theorem gives

$$|R_9| \le |a_{10}| = \frac{1}{10^2} = \frac{1}{100}.$$

15. (a) One unit vector that is parallel to $\mathbf{v} = [1, 2, -2]$ is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{[1, 2, -2]}{\sqrt{1+4+4}} = \left[\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right].$$

Another is $-\hat{\mathbf{v}} = \left[-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right]$.

(b) The vector (parallel) projection of $\mathbf{b} = [4, 2, 0]$ onto $\mathbf{a} = [1, -1, 1]$ is

$$\begin{aligned}
\text{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} \\
&= \left(\frac{4 - 2 + 0}{\sqrt{1+1+1}}\right) \frac{[1, -1, 1]}{\sqrt{3}} \\
&= \frac{2}{3}\,[1, -1, 1] = \left[\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right].
\end{aligned}$$