Student (Print) ______________________ Section ________

Last, First Middle

Student (Sign) ______________________

Student ID ______________________

Instructor ______________________

MATH 152, Fall 2007

Common Exam 3

Test Form B

Solutions

Instructions:

You may not use notes, books, calculator or computer.

Part I is multiple choice. There is no partial credit.

Mark the Scantron with a #2 pencil. For your own records, also circle your choices in this exam. Scantrons will be collected after 90 minutes and may not be returned.

Part II is work out. Show all your work. Partial credit will be given.

THE AGGIE CODE OF HONOR:
An Aggie does not lie, cheat or steal, or tolerate those who do.

For Dept use Only:	
1-11	/55
12	/15
13	/12
14	/12
15	/12
TOTAL	/106

Part I: Multiple Choice (5 points each)

There is no partial credit.

1. Compute $\lim\limits_{n\to\infty} \dfrac{2^n}{1+3^n}$

a. 0 correct choice

b. $\dfrac{2}{3}$

c. $\dfrac{1}{1-\frac{2}{3}}$

d. $\dfrac{\frac{1}{2}}{1-\frac{2}{3}}$

e. ∞

$$\lim_{n\to\infty} \frac{2^n}{1+3^n} = \lim_{n\to\infty} \frac{\frac{2^n}{3^n}}{\frac{1}{3^n}+1} = \frac{0}{1} = 0$$

2. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n}$

a. 15

b. 9

c. $\dfrac{15}{2}$

d. $\dfrac{9}{2}$ correct choice

e. The series diverges.

Geometric: $a = a_1 = \dfrac{3^{1+1}}{5^1} = \dfrac{9}{5}$ $\quad r = \dfrac{3}{5}$ (converges)

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n} = \frac{\frac{9}{5}}{1-\frac{3}{5}} = \frac{9}{5-3} = \frac{9}{2}$$

3. Compute $S = \displaystyle\sum_{n=1}^{\infty} (2^{1/n} - 2^{1/(n+1)})$

a. 3

b. 2

c. 1 correct choice

d. 0

e. The series diverges.

$$S_k = \sum_{n=1}^{k} (2^{1/n} - 2^{1/(n+1))}) = (2^1 - 2^{1/2}) + (2^{1/2} - 2^{1/3}) + \cdots + (2^{1/k} - 2^{1/(k+1)}) = 2^1 - 2^{1/(k+1)}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} (2^1 - 2^{1/(k+1)}) = 2^1 - 2^0 = 2 - 1 = 1$$

4. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is

a. absolutely convergent but not convergent.
b. convergent but not absolutely convergent. correct choice
c. absolutely convergent.
d. divergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent because it is an alternating decreasing series and $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.
The related absolute series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent because it is a p-series with $p = \frac{1}{2} < 1$. So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent but not absolutely convergent.

5. Find the coefficient of x^8 in the Maclaurin series for $f(x) = \cos(x^2)$.

a. 0
b. 1
c. $\frac{1}{2}$
d. $\frac{1}{6}$
e. $\frac{1}{24}$ correct choice

$\cos t = 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \cdots$ $\cos x^2 = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 + \cdots$ The coefficient of x^8 is $\frac{1}{24}$.

6. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ HINT: Think about a known Maclaurin series.

a. $-e^3$
b. $\frac{-1}{e^3}$
c. $\frac{1}{e^3}$ correct choice
d. $\frac{1}{3}$
e. The series diverges.

$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ So $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!} = e^{-3} = \frac{1}{e^3}$

7. Compute $\lim_{x\to 0} \dfrac{\sin(x^3) - x^3}{x^9}$

a. $-\dfrac{1}{6}$ correct choice

b. $-\dfrac{1}{24}$

c. $\dfrac{1}{24}$

d. $\dfrac{1}{6}$

e. $\dfrac{1}{3}$

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{\left[x^3 - \dfrac{(x^3)^3}{3!} + \cdots \right] - x^3}{x^9} = \lim_{x\to 0} \left[-\frac{(x^3)^3}{3!x^9} + \cdots \right] = -\frac{1}{6}$$

8. The series $S = \sum_{n=0}^{\infty} \dfrac{2^n}{1+3^n}$ satisfies

a. $S = 0$

b. $0 < S < 3$ correct choice

c. $S = 3$

d. $S > 3$

e. The series diverges.

$\sum_{n=0}^{\infty} \dfrac{2^n}{3^n}$ is geometric. $a = 1$ and $r = \dfrac{2}{3}$.

Since $|r| < 1$, $\sum_{n=0}^{\infty} \dfrac{2^n}{3^n} = \dfrac{1}{1 - \frac{2}{3}} = 3$ So $\sum_{n=0}^{\infty} \dfrac{2^n}{1+3^n} < \sum_{n=0}^{\infty} \dfrac{2^n}{3^n} = 3$

Also $\sum_{n=0}^{\infty} \dfrac{2^n}{1+3^n} > 0$ because all terms are positive.

9. Find the equation of the sphere centered at $(1,2,3)$ which passes through the point $(3,0,2)$.

a. $(x-3)^2+(y)^2+(z+1)^2 = 3$
b. $(x+3)^2+(y)^2+(z-1)^2 = 3$
c. $(x+3)^2+(y)^2+(z-1)^2 = 9$
d. $(x-1)^2+(y-2)^2+(z-3)^2 = 3$
e. $(x-1)^2+(y-2)^2+(z-3)^2 = 9$ correct choice

The radius is the distance from $(1,2,3)$ to $(3,0,2)$:

$$r = d\big((1,2,3),(3,0,2)\big) = \sqrt{(1-3)^2+(2-0)^2+(3-2)^2} = \sqrt{2^2+2^2+1^2} = 3$$

The circle is $(x-1)^2+(y-2)^2+(z-3)^2 = 9$

10. A vector $\vec{u}$ has length 5. A vector $\vec{v}$ has length 4. The angle between them is $45°$. Find $\vec{u}\cdot\vec{v}$.

a. 20
b. $20\sqrt{2}$
c. 10
d. $10\sqrt{2}$ correct choice
e. 5

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta = 5\cdot 4\cdot\cos 45° = \frac{20}{\sqrt{2}} = 10\sqrt{2}$$

11. A triangle has vertices $A=(1,2,3)$, $B=(2,3,3)$ and $C=(1,3,2)$. Find the angle at A.

a. $90°$
b. $60°$ correct choice
c. $45°$
d. $30°$
e. $0°$

$$\overrightarrow{AB} = B-A = (2,3,3)-(1,2,3) = (1,1,0) \qquad \overrightarrow{AC} = C-A = (1,3,2)-(1,2,3) = (0,1,-1)$$

$$\overrightarrow{AB}\cdot\overrightarrow{AC} = 0+1+0 = 1 \qquad \left|\overrightarrow{AB}\right| = \sqrt{1+1+0} = \sqrt{2} \qquad \left|\overrightarrow{AC}\right| = \sqrt{0+1+1} = \sqrt{2}$$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{1}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2} \qquad \theta = 60°$$

Part II: Work Out (points indicated)

Show all your work. Partial credit will be given.

12. (15 points) For each series, determine if it is convergent or divergent.
Be sure to identify the Convergence Test and check out its hypotheses.

a. $\displaystyle\sum_{n=2}^{\infty} n e^{\left(-n^2\right)}$

Integral Test: Consider $f(x) = x e^{\left(-x^2\right)}$

Note: The terms are positive, f interpolates: $f(n) = n e^{\left(-n^2\right)}$ and

f is decreasing because $f'(x) = e^{\left(-x^2\right)} - x e^{\left(-x^2\right)} 2x = e^{\left(-x^2\right)}(1 - 2x^2) < 0$ for $x \geq 2$.

We integration using the substitution $u = -x^2$ $du = -2x\,dx$

$\displaystyle\int_2^{\infty} x e^{\left(-x^2\right)}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2} e^{\left(-x^2\right)}\Big|_2^{\infty} = \frac{1}{2} e^{-4}$ converges

So $\displaystyle\sum_{n=2}^{\infty} n e^{\left(-n^2\right)}$ converges.

b. $\displaystyle\sum_{n=2}^{\infty} \frac{n^4 - 2}{n^5 + 1}$

Limit Comparison Test:

Compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ which is a divergent harmonic series (p-series with $p = 1$).

Note: The Comparison Test will not work because $\dfrac{n^4 - 2}{n^5 + 1} < \dfrac{n^4}{n^5} = \dfrac{1}{n}$

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4 - 2}{n^5 + 1} \cdot \frac{n}{1} = 1$ which is finite and non-zero.

So $\displaystyle\sum_{n=2}^{\infty} \frac{n^4 - 2}{n^5 + 1}$ diverges also.

c. $\sum_{n=2}^{\infty}(-1)^n \frac{n+1}{n-1}$

(N-th Term) Test for Divergence: $\lim_{n\to\infty} \frac{n+1}{n-1} = 1$

So $(-1)^n \frac{n+1}{n-1}$ oscillates between numbers close to 1 and -1.

So $\lim_{n\to\infty}(-1)^n \frac{n+1}{n-1}$ does not exist.

So $\sum_{n=2}^{\infty}(-1)^n \frac{n+1}{n-1}$ diverges.

13. (12 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x-5)^n$.

Ratio Test: $|a_n| = \frac{|x-5|^n}{n3^n}$ $|a_{n+1}| = \frac{|x-5|^{n+1}}{(n+1)3^{n+1}}$

$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-5|^{n+1}}{(n+1)3^{n+1}} \frac{n3^n}{|x-5|^n} = \frac{|x-5|}{3} \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \frac{|x-5|}{3}$

The series is absolutely convergent when $\rho = \frac{|x-5|}{3} < 1$ or $|x-5| < 3$.

So the radius of convergence is $R = 3$ and the series is absolutely convergent on $(2,8)$.

We check the endpoints:

$x = 2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(2-5)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent harmonic series.

$x = 8$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(8-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent by the Alternating Series Test.

So the interval of convergence is $2 < x \le 8$ or $(2,8]$.

14. (12 points)

a. Find the 2nd degree Taylor polynomial, $T_2(x)$, for $f(x) = \frac{1}{\sqrt{x}}$ at $x = 4$.

$$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \qquad f'(x) = -\frac{1}{2}x^{-3/2} = \frac{-1}{2\sqrt{x}^3} \qquad f''(x) = \frac{3}{4}x^{-5/2} = \frac{3}{4\sqrt{x}^5}$$

$$f(4) = \frac{1}{\sqrt{4}} = \frac{1}{2} \qquad f'(4) = \frac{-1}{2\sqrt{4}^3} = \frac{-1}{16} \qquad f''(4) = \frac{3}{4\sqrt{4}^5} = \frac{3}{128}$$

$$T_2(x) = f(4) + f'(4)(x-4) + \frac{1}{2}f''(4)(x-4)^2 = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2$$

b. Use this polynomial to approximate $\frac{1}{\sqrt{6}}$. Do not simplify.

$$\frac{1}{\sqrt{6}} = f(6) \approx T_2(6) = \frac{1}{2} - \frac{1}{16}(6-4) + \frac{3}{256}(6-4)^2$$ OK to stop here

$$= \frac{1}{2} - \frac{1}{8} + \frac{3}{64} = 0.42188$$

Exact value: $\frac{1}{\sqrt{6}} = 0.40825$

c. Estimate the error in your approximation in part (b). Do not simplify. Justify your answer.

METHOD 1: This is the beginning of an alternating series.
So the error in the approximation is less than the absolute value of the next term.

$$f'''(x) = \frac{-15}{8}x^{-7/2} = \frac{-15}{8\sqrt{x}^7} \qquad f'''(4) = \frac{-15}{8\sqrt{4}^7} = \frac{-15}{1024}$$

The cubic term in T_3 is $\frac{1}{6}f'''(4)(x-4)^3 = \frac{1}{6}\frac{-15}{1024}(x-4)^3 = \frac{-5}{2048}(x-4)^3$.

The next term in $\frac{1}{\sqrt{6}}$ is obtained by setting $x = 6$ or $\frac{-5}{2048}(6-4)^3 = \frac{-5}{256}$

$$|E| < \frac{5}{256} \approx 1.95 \times 10^{-2}$$

METHOD 2: The Taylor Remainder Theorem says:
If you use $T_n(x)$, the Taylor polynomial of degree n, to approximate the function, $f(x)$, then the remainder, $R_n(x) = f(x) - T_n(x)$, is bounded by $|R_n(x)| \le \frac{M(x-a)^{n+1}}{(n+1)!}$
where a is the center and $M \ge f^{(n+1)}(c)$ for all c between x and a.

In this case, $n = 2$, $a = 4$, $x = 6$ and $f'''(x) = \frac{-15}{8}x^{-7/2} = \frac{-15}{8\sqrt{x}^7}$.

On the interval $[4,6]$, the largest value of $\frac{15}{8\sqrt{x}^7}$ occurs at $x = 4$.

So we take $M = \frac{15}{8\sqrt{4}^7} = \frac{15}{1024}$. Then $|R_2(x)| \le \frac{15}{1024}\frac{(6-4)^3}{(3)!} = \frac{5}{256}$.

15. (12 points)

a. Find a Maclaurin series for e^{-x^2}.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \cdots \quad \text{(alternating series)}$$

b. Write the integal $\int_0^{0.1} e^{-x^2}\, dx$ as an infinite series using summation notation.

$$\int_0^{0.1} e^{-x^2}\, dx = \int_0^{0.1} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \right) dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} \right]_0^{0.1} = \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{2n+1}}{n!\,(2n+1)}$$

$$= 0.1 - \frac{10^{-3}}{3} + \frac{10^{-5}}{10} - \frac{10^{-7}}{42} + \cdots \quad \text{(alternating series)}$$

c. Use the series from part (b) to approximate the integal $\int_0^{0.1} e^{-x^2}\, dx$ to within $\pm 10^{-5}$. Explain how you know your estimate is correct to within $\pm 10^{-5}$.

Keeping 2 terms:

$$\int_0^{0.1} e^{-x^2}\, dx \approx 0.1 - \frac{.001}{3} \approx 0.1 - .0003333 = .0996667$$

The integral is expressed as an alternating series.

When an alternating series is approximated by using the first k terms

the error is less than the absolute value of the $(k+1)$-th term.

In this case the error is less than $\frac{10^{-5}}{10} = 10^{-6} < 10^{-5}$.