Part 1 Multiple Choice (45 points)

Read each question carefully; each problem is worth 5 points. Mark your responses on the ScanTron form and on the exam itself

1. Suppose that f is a continuous function defined on the interval $[0, 1]$. Given below is a table of values of f:

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$	1	1	0	$-\frac{1}{2}$	2

Use the data above, along with the *Trapezoidal Rule* with $n = 4$, to compute an approximate value of $\int_0^1 f(x)\,dx$.

   a. 1

   b. $\frac{1}{8}$

   c. 0

   d. $\frac{1}{2}$ correct choice

   e. $\frac{1}{4}$

$$T_4 = \Delta x\left(\frac{1}{2}f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + \frac{1}{2}f(1)\right) = \frac{1}{4}\left(\frac{1}{2} + 1 + 0 - \frac{1}{2} + \frac{1}{2}2\right) = \frac{1}{2}$$

2. Let f be the function given in the previous question. Use *Simpson's Rule* with $n = 4$ to compute an approximate value of $\int_0^1 f(x)\,dx$.

   a. $\frac{5}{4}$

   b. $\frac{5}{12}$ correct choice

   c. 1

   d. 0

   e. $\frac{1}{12}$

$$S_4 = \frac{\Delta x}{3}\left(f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right) = \frac{1}{12}(1 + 4 + 0 - 2 + 2) = \frac{5}{12}$$

3. Evaluate the improper integral $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\, dx$.

a. 1

b. 2   correct choice

c. $\frac{1}{2}$

d. $+\infty$

e. $-\infty$

$$u = \sin x \qquad du = \cos x\, dx \qquad I = \int_0^1 \frac{du}{\sqrt{u}} = 2\sqrt{u}\,\big|_0^1 = 2$$

4. Compute the length of the parametric curve given by the equations
$$x(t) = 1 - 2\cos t, \qquad y(t) = 2\sin t, \qquad 0 \le t \le \pi.$$

a. 2

b. 2π correct choice

c. 4

d. 4π

e. π

$$L = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^\pi \sqrt{(2\sin t)^2 + (2\cos t)^2}\, dt = \int_0^\pi 2\, dt = 2\pi$$

5. Let C denote the curve $y = e^{x^2}$, $0 \le x \le 1$. Which of the following integrals gives the area of the surface obtained by rotating C about the x-axis?

a. $2\pi \int_0^1 \sqrt{1 + 4x^2 e^{2x^2}}\, dx$

b. $2\pi \int_0^1 e^{x^2} \sqrt{1 + e^{2x^2}}\, dx$

c. $2\pi \int_0^1 e^{x^2} \sqrt{1 + 4x^2 e^{x^4}}\, dx$

d. $2\pi \int_0^1 e^{x^2} \sqrt{1 + 4x^2 e^{2x^2}}\, dx$ correct choice

e. $2\pi \int_0^1 e^{x^2} \sqrt{1 + e^{x^4}}\, dx$

$$A = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int 2\pi e^{x^2} \sqrt{1 + \left(e^{x^2} 2x\right)^2}\, dx = 2\pi \int_0^1 e^{x^2} \sqrt{1 + 4x^2 e^{2x^2}}\, dx$$

6. Given that $\frac{dy}{dx} = xy$, and $y(0) = 1$, determine y.

    a. $y = e^{2x}$

    b. $y = e^{2x^2}$

    c. $y = e^{x^2}$

    d. $y = e^x$

    e. $y = e^{x^2/2}$ correct choice

$$\int \frac{dy}{y} = \int x\,dx \qquad \ln|y| = \frac{x^2}{2} + C \qquad |y| = e^C e^{x^2/2} \qquad y = Ae^{x^2/2}$$
$$x = 0, \quad y = 1 \qquad 1 = Ae^0 \qquad A = 1 \qquad y = e^{x^2/2}$$

7. Find an integrating factor $I(x)$ for the differential equation $y' + (\tan^2 x)y = x$, in the interval $0 < x < \frac{\pi}{2}$.

    a. $I(x) = e^{\sec^2 x}$

    b. $I(x) = \sec^2 x$

    c. $I(x) = e^{\tan x - x}$ correct choice

    d. $I(x) = \tan x - x$

    e. $I(x) = e^{2\tan x \sec^2 x}$

$$P = \tan^2 x \qquad \int P\,dx = \int \sec^2 x - 1\,dx = \tan x - x \qquad I = e^{\int P\,dx} = e^{\tan x - x}$$

8. Decide on the convergence/divergence of each of the following improper integrals:

$$\text{(I)} \quad \int_1^\infty \frac{1}{x^3}\,dx \qquad \text{(II)} \quad \int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad \text{(III)} \quad \int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$$

    a. (III) is convergent, (I) and (II) are divergent.

    b. (II) is convergent, (I) and (III) are divergent.

    c. (I) is convergent, (II) and (III) are divergent.

    d. (I) and (II) are convergent, (III) is divergent.

    e. (I) and (III) are convergent, (II) is divergent.   correct choice

(I) $\int_1^\infty \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2} \right]_1^\infty = 0 - -\frac{1}{2} = \frac{1}{2}$ converges

(II) $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = [2\sqrt{x}]_1^\infty = \infty - 2 = \infty$ diverges

(III) $\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx < \int_1^\infty e^{-x}\,dx = [-e^{-x}]_1^\infty = -\lim_{x \to \infty} e^{-x} - -e^{-1} = e^{-1}$ converges

9. Suppose that f is a continuously differentiable function defined on the interval $[0,1]$, and let $g(x) = 2f\left(\frac{x}{2}\right)$, $0 \le x \le 2$. Given that the length of the curve $y = f(x)$, $0 \le x \le 1$, is 3, determine the length of the curve $y = g(x)$, $0 \le x \le 2$.

   a. 3

   b. 6   correct choice

   c. $\frac{3}{2}$

   d. $\frac{2}{3}$

   e. insufficient information to make a determination

$$g' = 2f'\left(\frac{x}{2}\right)\frac{1}{2} = f'\left(\frac{x}{2}\right) \qquad L = \int_0^2 \sqrt{1 + g'(x)^2}\, dx = \int_0^2 \sqrt{1 + f'\left(\frac{x}{2}\right)^2}\, dx$$

$$u = \frac{x}{2} \qquad du = \frac{1}{2}\, dx \qquad dx = 2\, du \qquad L = \int_0^1 \sqrt{1 + f'(u)^2}\, 2\, du = 2 \cdot 3 = 6$$

Part 2 (60 points)

Present your solutions to the following problems (10-14) in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.
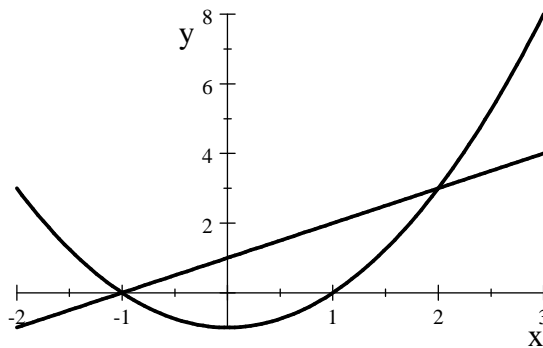
10. (12 points) Determine the x co-ordinate of the centroid of the region enclosed by the parabola $y = x^2 - 1$ and the straight line $y = x + 1$.

$$x^2 - 1 = x + 1$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1, 2$$

$$A = \int_{-1}^{2} (x + 1) - (x^2 - 1)\, dx = \int_{-1}^{2} x + 2 - x^2\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$
$$= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) = \frac{9}{2}$$

$$A_y = \int_{-1}^{2} x(x + 2 - x^2)\, dx = \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4} \right]_{-1}^{2} = \left(\frac{8}{3} + 4 - 4 \right) - \left(\frac{-1}{3} + 1 - \frac{1}{4} \right) = \frac{9}{4}$$

$$\bar{x} = \frac{A_y}{A} = \frac{9}{4}\frac{2}{9} = \frac{1}{2}$$

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11. (12 points) Compute the area of the surface obtained by rotating the curve $x = 2\sqrt{1-y}$, $0 \le y \le 1$, about the y-axis.

$$\frac{dx}{dy} = \frac{-2}{2\sqrt{1-y}} = \frac{-1}{\sqrt{1-y}}$$

$$A = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = 4\pi \int_0^1 \sqrt{1-y}\sqrt{1 + \frac{1}{1-y}}\, dy = 4\pi \int_0^1 \sqrt{1-y+1}\, dy$$

$$= 4\pi \int_0^1 \sqrt{2-y}\, dy = 4\pi \left[-\frac{2(2-y)^{3/2}}{3} \right]_0^1 = 4\pi\left(-\frac{2}{3}\right) - 4\pi\left(-\frac{2(2)^{3/2}}{3}\right) = \frac{8\pi}{3}\left(2\sqrt{2} - 1\right)$$

12. (12 points) A tank is full of water. The end of the tank is vertical and has the shape of a circle of radius 10 m. Calculate the hydrostatic force against the end of the tank. (The density of water, ρ, is 1000 kg/m^3, and the acceleration due to gravity, g, is 9.8 m/s^2.)

Put the zero of the y-axis at the center of the circle and measure y upward.
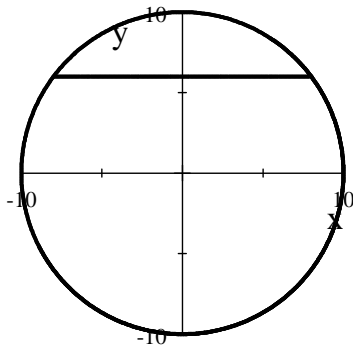
$$W = 2x = 2\sqrt{100 - y^2}$$
$$dA = W\, dy = 2\sqrt{100 - y^2}\, dy$$

$$F = \int \rho g h\, dA = \rho g \int_{-10}^{10} (10 - y) 2\sqrt{100 - y^2}\, dy = 20\rho g \int_{-10}^{10} \sqrt{100 - y^2}\, dy - \rho g \int_{-10}^{10} 2y\sqrt{100 - y^2}\, dy$$

The first integral is the area of a semicircle of radius 10.

In the second integral make the substitution $u = 100 - y^2$ $du = -2y\, dy$.

$$F = 20\rho g \left[\frac{1}{2}\pi(10)^2\right] + \rho g \int \sqrt{u}\, du = \rho g 1000\pi + \rho g \frac{2u^{3/2}}{3}$$

$$= \rho g 1000\pi + \rho g \left[\frac{2(100 - y^2)^{3/2}}{3}\right]_{-10}^{10} = \rho g 1000\pi = 9.8 \times 10^6 \pi$$

13. (12 points) Evaluate the improper integral

$$\int_1^\infty \frac{dx}{x(2x+1)}.$$

Show all your steps clearly and concisely.

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1} \qquad 1 = A(2x+1) + Bx \qquad x = 0:\ A = 1 \qquad x = -\frac{1}{2}:\ B = -2$$

$$\int_1^\infty \frac{dx}{x(2x+1)} = \int_1^\infty \left(\frac{1}{x} + \frac{-2}{2x+1}\right) dx = [\ln|x| - \ln|2x+1|]_1^\infty = \left[\ln\left|\frac{x}{2x+1}\right|\right]_1^\infty$$

$$= \lim_{x\to\infty} \ln\left|\frac{x}{2x+1}\right| - \ln\left|\frac{1}{2+1}\right| = \ln\left(\lim_{x\to\infty} \frac{x}{2x+1}\right) - \ln\frac{1}{3}$$

$$= \ln\left(\frac{1}{2}\right) - \ln\frac{1}{3} = \ln\left(\frac{3}{2}\right)$$

14. (12 points) Solve the following initial value problem:

$$(x^2+1)\,y' - 2xy = (x^2+1), \qquad y(0) = 1.$$

$$y' - \frac{2x}{x^2+1}y = 1 \qquad P = \frac{-2x}{x^2+1} \qquad \int P\,dx = -\ln(x^2+1) \qquad I = e^{\int P\,dx} = e^{-\ln(x^2+1)} = \frac{1}{x^2+1}$$

$$\frac{1}{x^2+1}y' - \frac{2x}{(x^2+1)^2}y = \frac{1}{x^2+1} \qquad \left(\frac{1}{x^2+1}y\right)' = \frac{1}{x^2+1}$$

$$\frac{1}{x^2+1}y = \int \frac{dx}{x^2+1} = \arctan x + C$$

$$x = 0,\ \ y = 1: \qquad \frac{1}{0^2+1}1 = \arctan 0 + C \qquad C = 1$$

$$\frac{1}{x^2+1}y = \arctan x + 1$$

$$y = (x^2+1)\arctan x + x^2 + 1$$

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