Solutions to MATH 152 Fall 2008 Exam 3B

1. **A** $\lim_{n\to\infty}(a_n^2 - 3b_n) = (\lim_{n\to\infty} a_n)^2 - 3\lim_{n\to\infty} b_n = 2^2 + 3(-3) = -5.$

2. **E** Apply L'Hospital's Rule to $\lim_{x\to\infty}\frac{x}{\ln x} = \lim_{x\to\infty}\frac{1}{\frac{1}{x}} = \infty$

3. **E** Complete the square: $(x^2 - 2x + 1) + y^2 + z^2 = 2 + 1$; $(x-1)^2 + y^2 + z^2 = 3$, so $r^2 = 3$ and $r = \sqrt{3}$.

4. **B** Using the Comparison Test. (D) is NOT necessarily true because $\lim_{n\to\infty} a_n = 0$ does not necessarily mean that $\sum_{n=1}^{\infty} a_n$ is convergent.

5. **A** Since $\lim_{n\to\infty}\frac{n}{n+1} = 1$, the sequence $(-1)^n\frac{n}{n+1}$ alternates between -1 and 1, therefore the terms of the series do not approach 0, which means the series diverges by the Test for Divergence.

6. **D** Let $a_n = \frac{1}{n}$. Then $\lim_{n\to\infty}\frac{b_n}{a_n} = 1$, which means the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ either both converge or both diverge. Since $\sum_{n=1}^{\infty} a_n$ diverges (by Integral Test or P-Test), $\sum_{n=1}^{\infty} b_n$ diverges by the Limit Comparison Test.

7. **B** $\sum_{n=1}^{\infty}\frac{1}{(n+3)(n+4)} = \sum_{n=1}^{\infty}\left(\frac{1}{n+3} - \frac{1}{n+4}\right)$, which is a Telescoping Series:
$s_N = \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{N+3} - \frac{1}{N+4}\right) = \frac{1}{4}$ as $N \to \infty$.

8. **B** Since the terms of both series approach zero, both series converge by the Alternating Series Test. To test absolute convergence, we look at (IA)$\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$ and (IIA)$\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$. Both are P-series; in (IA), $P > 1$so the series converges, and in (IIA), $P < 1$ so the series diverges. Therefore, series (I) converges absolutely, and series (II) converges but not absolutely.

9. **A** The series can be written as $\sum_{n=1}^{\infty}\left(\frac{2}{9}\right)\left(\frac{2}{3}\right)^{n-1}$, which is a geometric series with $a = \frac{2}{9}$ and $r = \frac{2}{3}$. The sum of the series is $\frac{\frac{2}{9}}{1-\frac{2}{3}} = \frac{2}{3}$.

10. **E** The Maclaurin series for $\cos x$ is $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$. So the Maclaurin series for $\cos(x^2)$ is $\sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}$.

11. (a) $\overline{\mathbf{AB}} =< 2, 1, -2 >$, $\overline{\mathbf{BC}} =< 1, 0, 1 >$. $\overline{\mathbf{AB}}\cdot\overline{\mathbf{BC}} = (2)(1) + (1)(0) + (-2)(1) = 0$, so the sides are perpendicular to each other.

    (b) $|\overline{\mathbf{AB}}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, $|\overline{\mathbf{BC}}| = \sqrt{1^2 + 0 + 1^2} = \sqrt{2}$, so the area is $\frac{1}{2}|\overline{\mathbf{AB}}||\overline{\mathbf{BC}}| = \frac{3\sqrt{2}}{2}$.

12. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. Therefore, subtracting the first term of the series gives us $e^{-1} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

13. $f(-1) = 0$; $f'(x) = 8x^3 + 1$, so $f'(-1) = -7$; $f''(x) = 24x^2$, so $f''(-1) = 24$; $f'''(x) = 48x$, so $f'''(-1) = -48$. The third degree Taylor Polynomial is $f(-1) + \frac{f'(-1)}{1!}(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 = -7(x+1) + 12(x+1)^2 - 8(x+1)^3$.

14. (a) Applying the Ratio Test gives us absolute convergence when $\lim_{n\to\infty} \left| \frac{\frac{(x-1)^{n+1}}{3^{n+1}\sqrt{n+1}}}{\frac{(x-1)^n}{3^n\sqrt{n}}} \right| = \lim_{n\to\infty} \frac{|x-1|}{3} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} < 1$ and divergence when the limit is > 1. Since the second fraction approaches 1, we have absolute convergence when $\frac{|x-1|}{3} < 1$, $|x-1| < 3$ which makes the radius of convergence 3.

(b) The series converges when $-3 < x - 1 < 3$, $-2 < x < 4$. To find the interval of convergence, test the endpoints: When $x = -2$, the series becomes $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series test. When $x = 4$, the series becomes $\sum_{n=1}^{\infty} \frac{3^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges by the P-test or integral test. Therefore, the interval of convergence is $-2 \le x < 4$.

15. (a) $\int S(x)\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n}\,dx = \sum_{n=1}^{\infty} \int \frac{(-1)^{n+1}}{(2n+1)!} x^{2n}\,dx = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} x^{2n+1}$

(b) $\int_0^{1/2} S(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} x^{2n+1}\Big|_0^{1/2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} \left(\frac{1}{2}\right)^{2n+1}$.

(c) Since the series is alternating $|S - S_3| \le |a_4| = \frac{1}{(2\cdot 4+1)(2\cdot 4+1)!} \left(\frac{1}{2}\right)^{2\cdot 4+1} = \frac{1}{9\cdot 9!} \left(\frac{1}{2}\right)^9$.